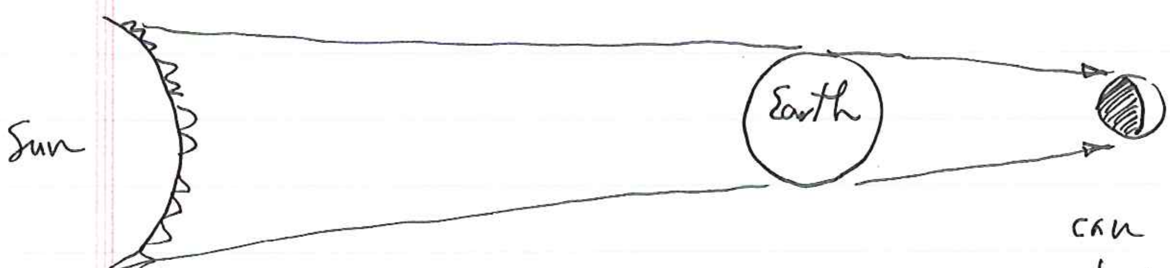
Lecture #1 : Earth – Third Rock from the Sun

Gross properties of Earth (and other planets)
Size, mass, mean density, ...

On a local scale, e.g. in a ship far out at sea, the $\oplus$ looks flat.

Spherical shape of $\oplus$ known to Greeks, e.g. Aristotle 4th century BC.
Correct interpretation of lunar eclipse = shadow of $\oplus$ on moon



can see Earth's shadow move across the moon

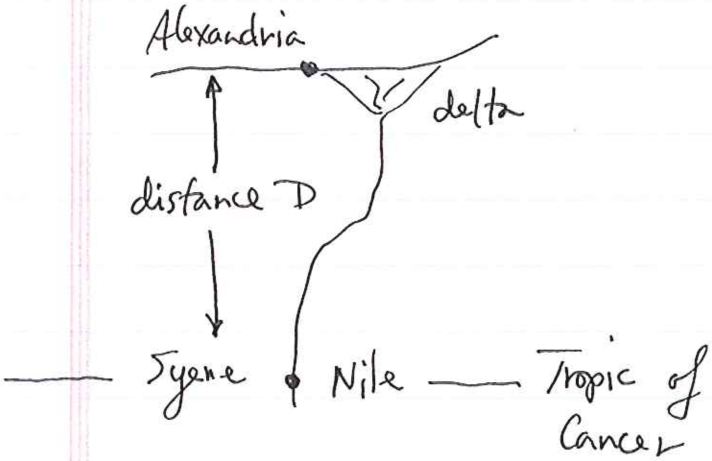
Size of Earth (radius) first measured in 3rd century BC by another Greek, Eratosthenes, who was in charge of the famous library at Alexandria, ~ 225 BC

Prominent astronomer & geographer
Developed ideas of latitude & longitude

His map of the known world set the standard for the classical era.

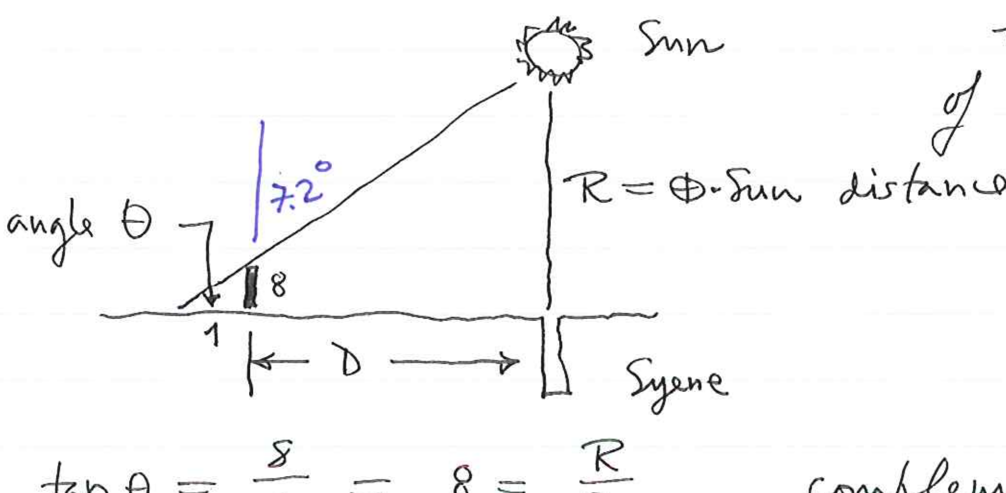
Observed that on summer solstice (longest day of year) sun directly overhead at Syene (near Aswan) on Tropic of Cancer, whereas at Alexandria, almost due north, it cast a shadow.

~~2nd century BC~~



Show location on Eratosthenes' own map — actually Ptolemy's but based on work of Eratosthenes

Flat earth interpretation of observation

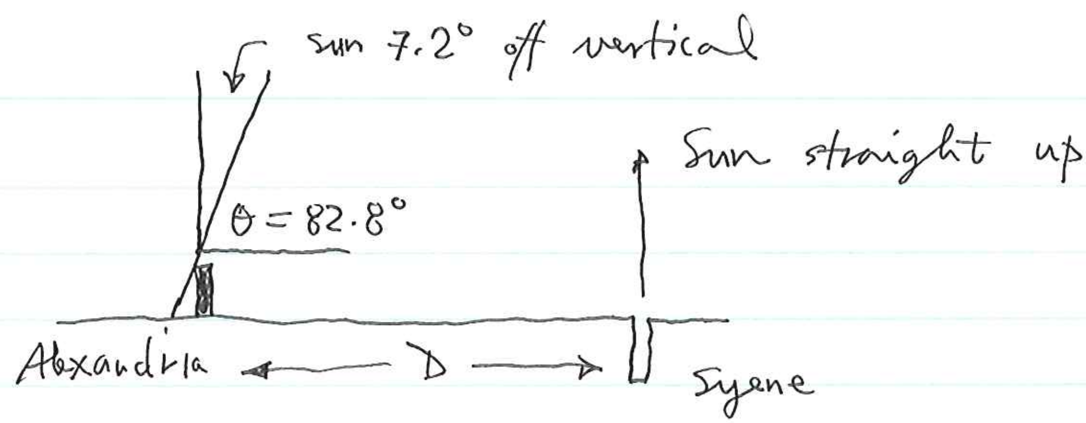


$$\tan\theta = \frac{8}{1} = 8 = \frac{R}{D}$$

complementary triangles

$$\boxed{R = 8D}$$

$$\theta = \arctan 8 = 82.8^\circ$$



He estimated the distance D from the time it took for a camel caravan to travel between them

$$D = 5000 \text{ stadia}$$

Stadium = length between start and finish line : controversy among scholars as to exact equivalent

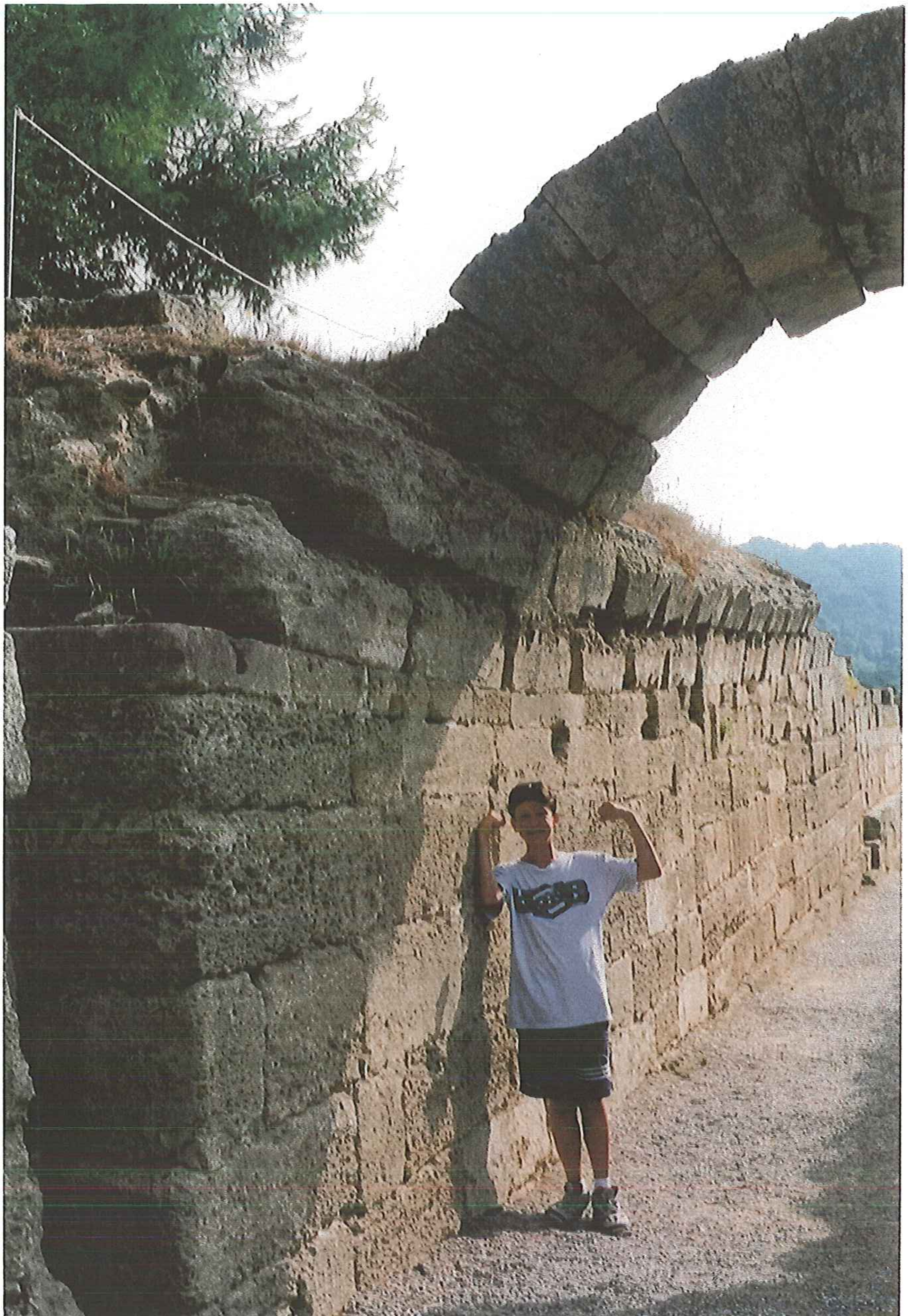
one stadium = 160 meters

5000 stadia = 800 km

$$\boxed{R = 8D = 6400 \text{ km}}$$

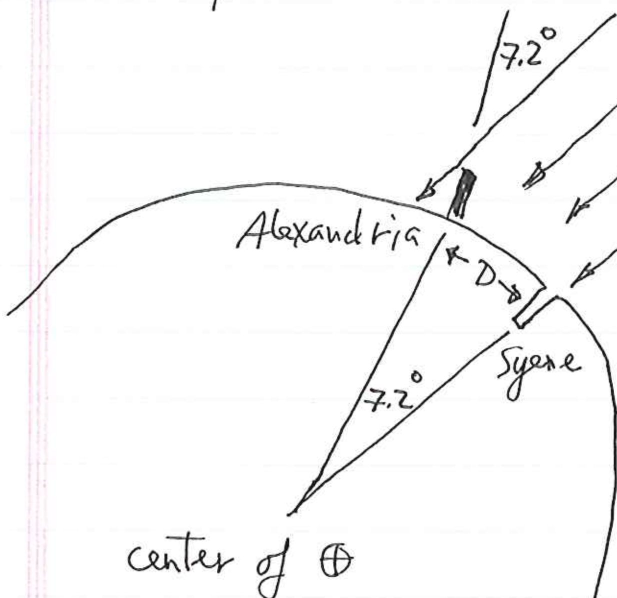
In the flat-earth view this would be the earth-sun distance.

But Eratosthenes was familiar with Aristotle's eclipse argument — rejected the flat-earth view.



Argued instead that the Sun was very distant >>> ⊕ dimension.

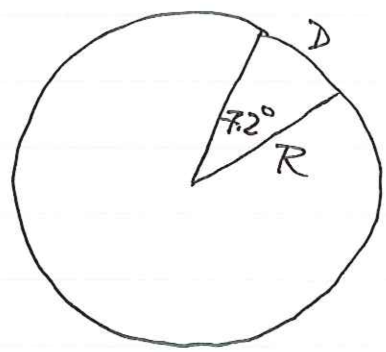
The picture then looks like this:



Sun so far away that rays of light appear parallel

$$\boxed{\frac{7.2^\circ}{360^\circ} = \frac{1}{50} \text{ of circle}}$$

i.e.



sphere of radius R

circumference $= 2\pi R$

$$D = \frac{7.2^\circ}{360^\circ} \cdot 2\pi R \quad \text{or}$$

$$\boxed{R = \frac{50}{2\pi} D = 8D}$$

$$\boxed{R = \frac{360}{2\pi\,(7.2)}\, D = 8D}$$

same answer as before, but interpretation very different

$$\boxed{R = \text{radius of } \oplus = 6400 \text{ km}}$$

the work of the polymath Eratosthenes (ca. 275–194 B.C.). His was a lasting contribution to the development of mapping, and with some justification he has been variously assigned a founding role in geography, cartography, and geodesy.[41] Although we are acquainted with his contribution only through later writers rather than through his original texts,[42] it is absolutely clear that in two scientific endeavors he surpassed both his predecessors and his contemporaries. The first of these was his measurement of the circumference of the earth, which was methodologically simple but brilliant.[43] The second was his construction of a world map based on both parallels and meridians, which was of seminal importance not only in the subsequent development of map projections but also in the eventual scientific and practical use of maps. Such a cartographic invention was equally applicable in chorographical or regional mapping and in geographical or world mapping, so that its key significance for the history of the map needs to be fully described.

Eratosthenes was born a Greek, in Cyrene (North Africa); going to Athens as a young man, he took lessons at one time from the Stoics and at another from the Academicians, among whom he was particularly influenced by Arcesilas of Pitane (Candarli), who had been a disciple of the mathematician Autolycus. In the work of Eratosthenes, as in that of his predecessors, the importance of his mastery of the geometry of the sphere and of the geocentric hypothesis cannot be overemphasized as providing the point of departure—as well as the theoretical framework—for the development of his cartographic ideas.[44] Eratosthenes' scientific distinction later attracted the attention of Ptolemy III Euergetes, king of Egypt 246–221 B.C. The king asked him to come to Alexandria as tutor to his son Philopator (born ca. 245 B.C.) and to take over the direction of the library when Apollonius left for Rhodes after adverse criticism of his poem *Argonautica*. At Alexandria, Eratosthenes was to compose two works on geographical subjects: one, *Measurement of the Earth*, explained the method used to find the circumference of the earth; the other, entitled *Geographica*, in three books, gave instructions for making a map of the inhabited world. Both works are lost, but Strabo, who begins his own work by a criticism of the *Geographica*, affords us fairly clear knowledge of its contents, and Cleomedes of the second century A.D. gives a brief summary of the *Measurement of the Earth*.[45]

From Cleomedes we learn that the method Eratosthenes used to evaluate the circumference of the earth was based on the geometry of the sphere.[46] According to the geocentric hypothesis, by which the earth was reduced to a point,[47] the sun's rays are parallel when falling on any point of the earth. It was known that Syene (Aswan) in Egypt was situated under the tropic; at midday on the summer solstice there was no shadow,

41. Cortesão, *History of Portuguese Cartography*, 1:78–79 (note 26); D. R. Dicks, "Eratosthenes," in *Dictionary of Scientific Biography*, 4:388–93 (note 30).

42. R. M. Bentham, "The Fragments of Eratosthenes of Cyrene" (typescript for Ph.D. thesis, University of London, 1948—author died before thesis was submitted); see also *Die geographischen Fragmente des Eratosthenes*, ed. Hugo Berger (Leipzig: Teubner, 1898). We know of Eratosthenes' *Geographica* mostly through Strabo.

43. Gerald R. Crone, *Maps and Their Makers: An Introduction to the History of Cartography*, 5th ed. (Folkestone, Kent: Dawson; Hamden, Conn.: Archon Books, 1978), 3.

44. In this respect Eratosthenes was heir to a continuous tradition of mathematical learning that can be traced back at least as far as Eudoxus, from whose day onward it is likely that treatises entitled *Sphaerica* had existed. Autolycus of Pitane (fl. 310 B.C.) was clearly a link in this chain of writers influencing Eratosthenes. Although Autolycus's textbook *On the Sphere in Motion*, composed about 330 B.C., was not an original work, it was a competent summary of a number of basic theorems concerning celestial phenomena for given places of observation, and it explained clearly the geometric relationship between the sky and the earth and the need for astronomical knowledge to define the position on the earth of any place of observation. See Autolycus of Pitane, *La sphère en mouvement*, ed. and trans. Germaine Aujac, Jean-Pierre Brunet, and Robert Nadal (Paris: Belles Lettres, 1979). It is also likely that the writings of Euclid (fl. Alexandria ca. 300 B.C.) were known to Eratosthenes. The *Elements* had been completed about 300 B.C., but Euclid was also the author of a small treatise entitled *Phaenomena*, which applied specifically to the celestial sphere the conclusions Autolycus drew for rotating spheres in general. After establishing the geometry of the rotating celestial sphere, Euclid examined the rising and setting of stars as a means of measuring time at night; to do this he had to analyze the relationship between the observer's horizon and the ecliptic on the celestial sphere, which is different for each parallel on the earth. A brief summary of this work is given by Pierre Chiron, "Les Phénomènes d'Euclide," in *L'astronomie dans l'antiquité classique*, Actes du Colloque tenu à l'Université de Toulouse–Le Mirail, 21–23 Octobre 1977 (Paris: Belles Lettres, 1979), 83–89. For early spherical astronomy, see Otto Neugebauer, *A History of Ancient Mathematical Astronomy* (New York: Springer-Verlag, 1975), 748–67.

45. Cleomedes *De motu circulari* (note 30). The original Greek title is Κυκλικὴ Θεωρία τῶν Μετεώρων.

46. Cleomedes *De motu circulari* 1.10 (note 30). An English translation of book 1, chap. 10, appears in Cortesão, *History of Portuguese Cartography*, 1:141–43 (note 26).

47. In the geocentric hypothesis, as explained in Euclid's *Phaenomena*, the sky of the fixed stars was compared to a sphere rotating around one diameter called the world axis. In the middle, the earth was reduced to a point that acted as center to the sphere; the fixed stars moved along parallel circles (being on a rotating sphere, they were all circles of the sphere perpendicular to the axis of rotation). The greatest of these parallel circles Euclid recognized as the celestial equator. But two other great circles were important: the oblique circle of the ecliptic (called the "zodiac" by Euclid; see Chiron, "Phénomènes d'Euclide," 85 [note 44]), and the circle of the visible horizon (the astronomical horizon dividing the visible celestial hemisphere from the invisible one), which remained motionless during the apparent motion of the celestial sphere (fig. 8.8). Euclid *Phaenomena* 1 and prop. 1, see Euclid, *Opera omnia*, 9 vols., ed. J. L. Heiberg and H. Menge (Leipzig: Teubner, 1883–1916), vol. 8, *Phaenomena et scripta musica* [*and*] *Fragmenta* (1916).

the sun being exactly at the zenith.[48] Supposing Alexandria to be on the same meridian as Syene (the difference is only 3°), Eratosthenes measured the angle between the direction of the sun and the vertical in Alexandria at midday on the summer solstice. This angle, one-fiftieth of a circle, was equal to the angle at the earth's center subtended by the arc of the meridian defined by Syene and Alexandria. Estimating the distance between the two towns at roughly 5,000 stades, Eratosthenes calculated the total circumference as 250,000 stades (fig. 9.4). He later extended the value to 252,000 so as to make it divisible by sixty.[49]

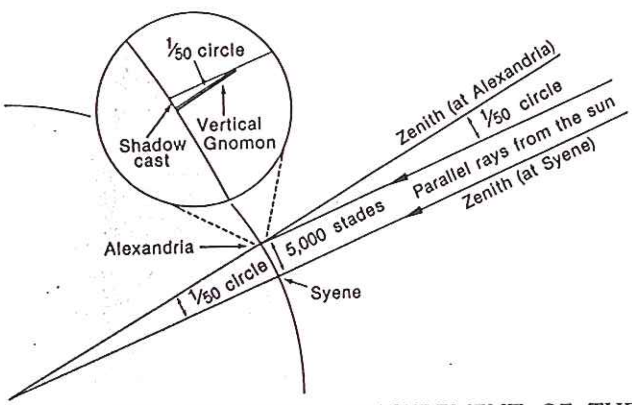


FIG. 9.4. ERATOSTHENES' MEASUREMENT OF THE EARTH. Eratosthenes worked with four assumptions: that Syene was on the tropic (at the summer solstice, the sun was thus directly overhead); that both it and Alexandria were on the same meridian; that the distance between them was 5,000 stades; and that the sun's rays were parallel. He knew that the difference in latitude between Alexandria and Syene was equivalent to the angle between the sun's rays and the zenith at Alexandria. From the lengths of a vertical stick (gnomon) and its shadow, he calculated this angle to be one-fiftieth of a circle. Thus the earth's circumference was estimated at 250,000 stades.
After John Campbell, *Introductory Cartography* (Englewood Cliffs, N.J.: Prentice-Hall, 1984), fig. 1.7.

Eratosthenes' method for calculating the circumference of the earth was sound, but its reliability depended on the accuracy of his base measurements and other assumptions. The angular distance between the two cities is quite accurate (7°12′ instead of the actual 7°7′), but Syene is not directly on the Tropic of Cancer but about 35′ to the north (using the modern figure for the obliquity of the ecliptic), and Alexandria and Syene are not on the same meridian. Furthermore, the distance between Alexandria and Syene is given in stades, the value of which has sparked considerable debate[50] quite apart from the question of the empirical source of the distance thus recorded.[51] Regardless of the actual value for the stade that Eratosthenes used or of the distance he arrived at—he knew that the distance between the two cities was a very rough estimate, as was his evaluation of the terrestrial circumference—the importance of his calculation lies in its influence. It is probable that after he had measured the circumference of the earth, Eratosthenes henceforth first established any distance in latitude by astronomical means, or by reference to the geometry of the sphere (the distance between equator and tropic being fixed at four-sixtieths of the great circle, for instance), and then evaluated this distance in stades. Thus the distance between equator and tropic, which had never been measured by surveyors, was said to be 16,800 stades.

The availability of knowledge of this estimate of the earth's circumference had three outstanding consequences. First, it was now possible to work out through geometry the length of every parallel circle on the earth. The parallel of Athens, for example, was "less than two hundred thousand stadia in circuit."[52] Second, differences of latitude, found by gnomonic methods and expressed in fractions of the circle, could easily be converted into stades. Third, it was now also possible to define the size of the inhabited world and its position on the surface of the terrestrial globe.

This third issue—the size and location of the inhabited world—was of intense and continuing interest to the Greeks, and having devised a method to answer this question, Eratosthenes was to return to its exposition in his *Geographica*. This work in three books is known to us mainly through Strabo. It was intended to provide a review and solution of all known problems involved in drawing a map of the earth (*gē-graphein*) or, more precisely, a map of the inhabited world on the surface of the terrestrial globe.[53] Starting from the theoretical premise that the earth is spherical, albeit with "certain irregularities of surface,"[54] Eratosthenes located the inhabited world completely in the Northern Hemisphere occupying the northern half of the distance between the Tropic of Cancer and the equator and the entire distance

48. Strabo *Geography* 17.1.48 (note 10).

49. Eratosthenes divided the earth into sixtieths; the use of 360° comes with Hipparchus.

50. On the matter of the modern value of the stade, see note 3 above.

51. Cortesão, *History of Portuguese Cartography*, 1:82 (note 26), speculates that Egyptian cadastral surveys may have been available to Eratosthenes in his calculation of the distance between the two points of observation.

52. Eratosthenes, in Strabo *Geography* 1.4.6 (note 10).

53. Eratosthenes was the first author to attempt this. His work began with a short history of geographical science from the time of Homer and the first mapmakers.

54. Strabo *Geography* 1.3.3 (note 10).

between that tropic and the polar circle. He calculated its width from north to south along the meridian that runs through Meroë, Alexandria, and Rhodes, resulting in a distance of 38,000 stades. Strabo described the overall shape of the *oikoumene* as somewhat like a chlamys, a Macedonian cloak perhaps resembling the shape in fig. 9.5.[55] Its length from west to east, however, he determined in accordance with an established concept, that its length was more than double the known breadth.

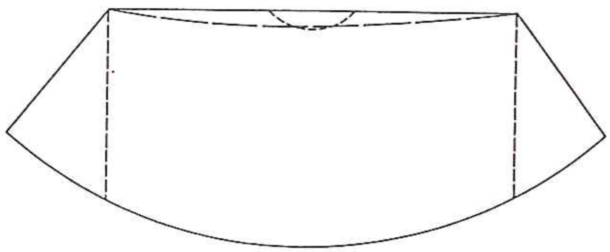


FIG. 9.5. THE CHLAMYS. This is the possible form of a common style of Macedonian cloak used by Strabo to illustrate the shape of the *oikoumene*. The top could be either straight or slightly curved.
Reconstructed from the description in *The Geography of Strabo*, 8 vols., ed. and trans. Horace Leonard Jones, Loeb Classical Library (Cambridge: Harvard University Press; London: William Heinemann, 1917–32), 2.5.6 and p. 435 n. 3.

Eratosthenes first described the distance from the capes of India to the extremities of Iberia as roughly 74,000 stades. Then (according to Strabo) Eratosthenes added 2,000 more stades to both west and east to keep the breadth from being more than half the length.[56] The total length thus became 78,000 stades.

The determination of the length of the inhabited world from India to Iberia was reckoned along the parallel of Athens. Eratosthenes believed this was less than 200,000 stades in circuit, "so that, if the immensity of the Atlantic Sea did not prevent, we could sail from Iberia to India along one and the same parallel over the remainder of the circle, that is, the remainder when you have subtracted the aforesaid distance, which is more than a third of the whole circle."[57] In fact, using a value for the circumference of the earth of 252,000 stades, 78,000 stades on the parallel in question is approximately equivalent to 138° of longitude, which is roughly the distance between the western coast of Spain and Korea rather than India.

It is not surprising that for many centuries to come values representing latitude were always much more reliable than those for longitude. Familiar with the geometry of the sphere, the Greeks were fairly well equipped to derive latitudes from direct observations of the sun and stars. In this respect, straightforward calculations could be undertaken to test the information of travelers. For longitudes the results were much less reliable, since it was necessary to observe an eclipse of the moon or other celestial body simultaneously from different places to obtain exact distances between them. Instead, the Greeks had to accept distances given by the itineraries without being able to verify them astronomically.

According to Strabo, it was in the third book of *Geographica* that Eratosthenes explained how to draw a map of the world:

> Eratosthenes, in establishing the map of the inhabited world, divides it into two parts by a line drawn from west to east, parallel to the equatorial line; and as ends of this line he takes, on the west, the Pillars of Heracles [Straits of Gibraltar], on the east, the capes and most remote peaks of the mountain-chain that forms the northern boundary of India. He draws the line from the Pillars through the Strait of Sicily [Straits of Messina] and also through the southern capes both of the Peloponnesus and of Attica, and as far as Rhodes and the Gulf of Issus [Gulf of Iskenderun, Turkey]; . . . then the line is produced in an approximately straight course along the whole Taurus Range as far as India, for the Taurus stretches in a straight course with the sea that begins at the Pillars, and divides all Asia lengthwise into two parts, thus making one part of it northern, the other southern; so that in like manner both the Taurus and the Sea from the Pillars up to the Taurus lie on the parallel of Athens.[58]

We can see from this passage that Eratosthenes had adopted the idea of the *diaphragma* (if not the term) introduced by Dicaearchus to divide the known world by means of a line parallel to the equator, drawn from west to east, beginning at the Straits of Gibraltar and running through Athens and Rhodes to India. It is also clear from other passages in Strabo that Eratosthenes drew a central perpendicular meridian through Rhodes, for he lists the places through which this passes and the distances between them.[59] Eratosthenes used very rough estimates and round numbers. The south-north distances (in stades) he provided between the following regions or towns were:

Between	*Stades*
Cinnamon country and Meroë	3,400
Meroë and Alexandria	10,000
Alexandria and Hellespont	about 8,100
Hellespont and river Borysthenes	5,000
River Borysthenes and parallel of Thule	about 11,500
Total	38,000

(Strabo *Geography* 1.4.2).

55. Strabo *Geography* 2.5.6 (note 10).
56. Strabo *Geography* 1.4.5 (note 10).
57. Eratosthenes, in Strabo *Geography* 1.4.6 (note 10).
58. Strabo *Geography* 2.1.1 (note 10).
59. Strabo *Geography* 2.5.42 (note 10).

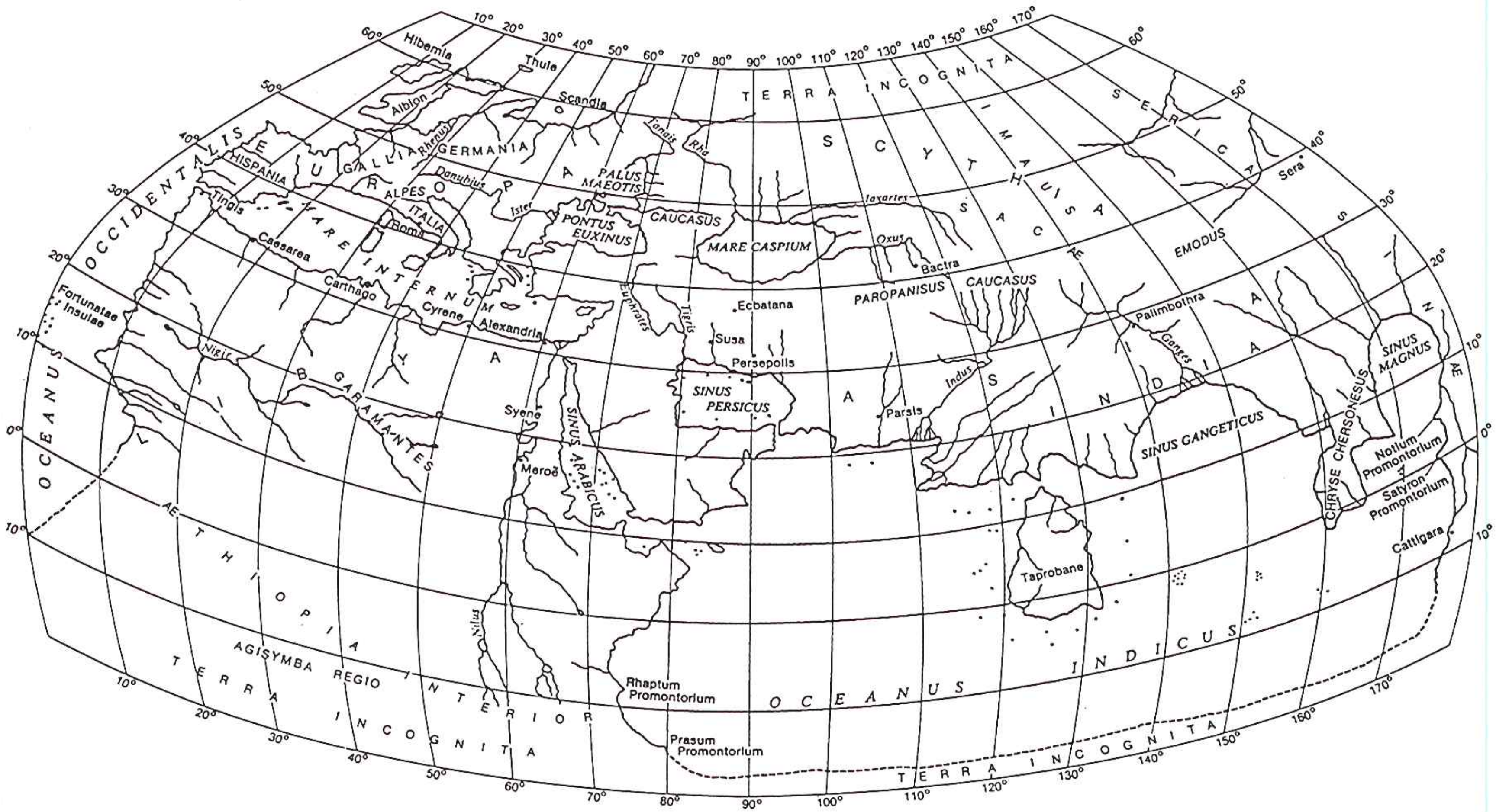


FIG. 11.3. A RECONSTRUCTION OF THE WORLD OF CLAUDIUS PTOLEMY. After Edward Herbert Bunbury, *A History of Ancient Geography among the Greeks and Romans from the Earliest Ages till the Fall of the Roman Empire*, 2d ed., 2 vols. (1883; republished with a new introduction by W. H. Stahl, New York: Dover, 1959), map facing p. 578.

Modern, more accurate measurement of mean radius

$$\boxed{R = 6371 \text{ km}}$$

Eratosthenes used his knowledge of the Earth's radius to construct a map of the known world.
Formed the basis of Ptolemy's better-known map.

Knowing the $\oplus$'s radius we can compute its volume.

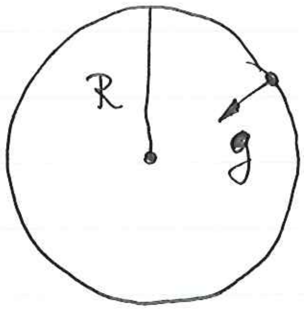
$$\boxed{\text{Volume of a sphere : } V = \frac{4}{3}\pi R^3}$$

$$\boxed{V_\oplus = 1.08 \cdot 10^{12} \text{ km}^3 = 1.08 \cdot 10^{21} \text{ m}^3}$$

The mass of the Earth was not accurately determined until much later.

Two ways to do this:

(1) measure acceleration of gravity at $\oplus$'s surface



Drop a mass — measure it's rate of acceleration — need an accurate fast clock

$$g = \frac{GM}{R^2}$$

M = mass of $\oplus$
G = Newton's constant

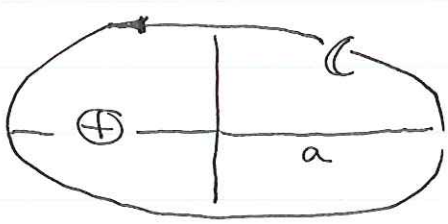
$$g = 9.8 \text{ m/sec}^2 \Rightarrow$$

$$GM_\oplus = 4 \cdot 10^{14} \text{ m}^3/\text{sec}^2$$

can now be measured to 10^{-9} or 10^{-10}

falling corner cube & atomic fountain

Method 2 — Kepler's third law



a = semimajor axis of ellipse
T = period of orbit $\approx$ 28 days

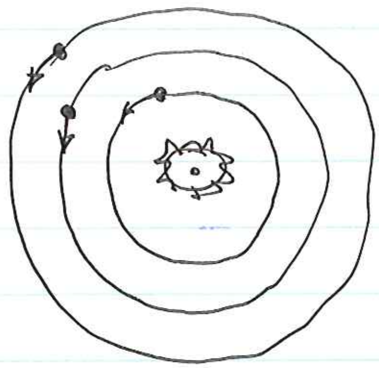
$$\frac{4\pi^2 a^3}{T^2} = G(M_\oplus + M_☾)$$

$$M_☾ \approx \frac{1}{81.3} M_\oplus$$

$$a \approx 60 R_\oplus$$

again gives $GM_\oplus = 4 \cdot 10^{14} \text{ m}^3/\text{sec}^2$

Kepler enunciated his law for the planetary orbits around the Sun



small

$$\frac{4\pi^2 a^3}{T^2} = G\left(M_{Sun} + \cancel{M_{planet}}\right)$$

$$\approx GM_{Sun} = \text{constant}$$

$$T^2 = \text{const } a^3$$

$$\boxed{T = \text{constant } a^{3/2}}$$

Showed that Tycho Brahe's data in good agreement with this empirical law.

Newton later provided a fundamental derivation of Kepler's 3 laws in terms of theory of gravitation.

Nowadays $GM_\oplus$ very accurately determined from artificial satellite orbits

$$\boxed{GM_\oplus = 3.986013 \cdot 10^{14} \text{ m}^3/\text{sec}^2}$$

Table 3-1. Characteristics of the orbits of the nine planets and of the largest object in the asteroid belt:

Planet	Radius of orbit (10^{13} centimeters)	Time for one revolution about Sun (years)	Inclination of orbit* (degrees)	Eccentricity of orbit†	Length of day (days)
Mercury	0.58	88 days 0.24	7.0	0.21	59
Venus	1.08	0.62	3.4	0.01	243
Earth	1.50	1.00	0.0	0.02	1.0
Mars	2.29	1.88	1.9	0.09	1.0
Asteroid Ceres	4.15	4.60	10.6	0.08	0.4
Jupiter	7.70	11.9	1.3	0.05	0.4
Saturn	14.3	29.5	2.5	0.06	0.4
Uranus	28.3	84.0	0.8	0.05	1.0
Neptune	45.1	164.8	1.8	0.01	0.9
Pluto	59.2	247.7	17.2	0.25	6.2

*The plane of the Earth's orbit is used as the reference.

†A measure of the deviation from circularity.

For all planets with moons, GM determined in same way.

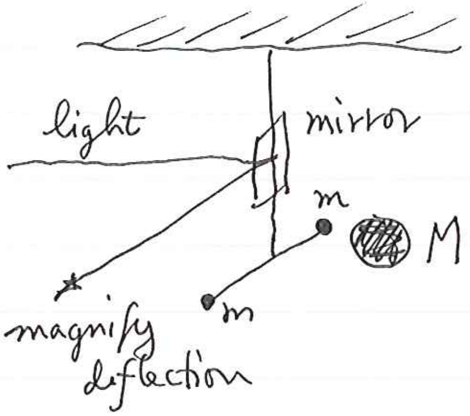
All geophysical measurements can only determine <u>G times M</u>.

To find M must measure G — first accurate measurement by Lord Cavendish 1798

$$\boxed{F = \frac{G M_1 M_2}{R^2}}$$

Gravitational force between two masses M_1 and M_2

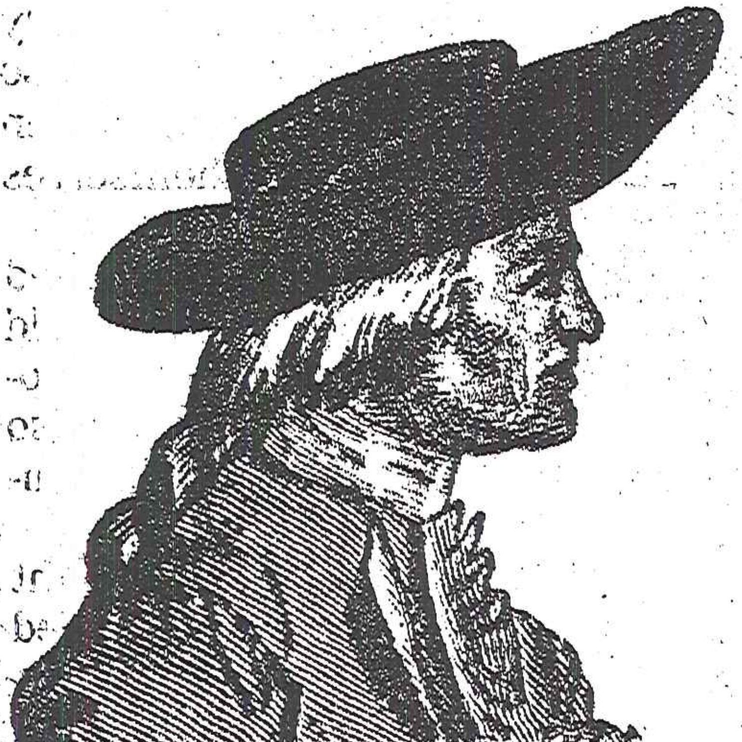
inverse-square law



move a big mass (shotput sized) up near a small mass M and measure deflection of torsion fibre

$$\boxed{G = 6.67 \cdot 10^{-11} \ m^3 \ kg^{-1} \ sec^{-2}}$$

Famous experiment — referred to as <u>weighing the Earth</u>



Corbis-Bettmann

Henry Cavendish, an 18th-century scientist, was known for his oddities.

Henry Cavendish discovered hydrogen, weighed the earth and investigated the powerful electric shocks delivered by the torpedo fish.

An 18th-century Englishman, Cavendish was known as widely for his oddity as for his accomplishments. Reclusive and detached, he published little, never married, dressed shabbily, appeared oblivious to his wealth and spoke rarely. He "probably uttered fewer words in the course of his life than any man who ever lived to fourscore years, not at all excepting the monks of La Trappe," a contemporary is said to have remarked.

Such traits, in the brilliant or highly creative, are often dismissed as simple eccentricity. But writing this month in the journal Neurology, the neurologist and author Oliver Sacks suggests that Cavendish may have had Asperger's syndrome, a psychiatric disorder thought to be related to autism.

$$\boxed{M_\oplus = 5.98 \cdot 10^{24} \text{ kg}}$$

Mean density of $\oplus$:

$$\boxed{\bar{\rho}_\oplus = \frac{M_\oplus}{V_\oplus} = \frac{5.98 \cdot 10^{24}}{1.08 \cdot 10^{21}} = 5500 \text{ kg/m}^3}$$

For comparison:

$\rho_{water} = 1000$ kg/m^3 at 20°C
1 atm

$$\boxed{\text{Specific gravity} = \frac{\text{density of substance}}{\text{density of water}}}$$

$$\boxed{\text{Mean } \oplus : \quad SG = 5.5}$$

About twice as heavy as that of common rocks collected on surface

$$\boxed{\begin{aligned} SG_{rocks} &= 2.5 - 3.3 \\ \rho_{rocks} &\simeq 2500 - 3300 \text{ kg/m}^3 \end{aligned}}$$

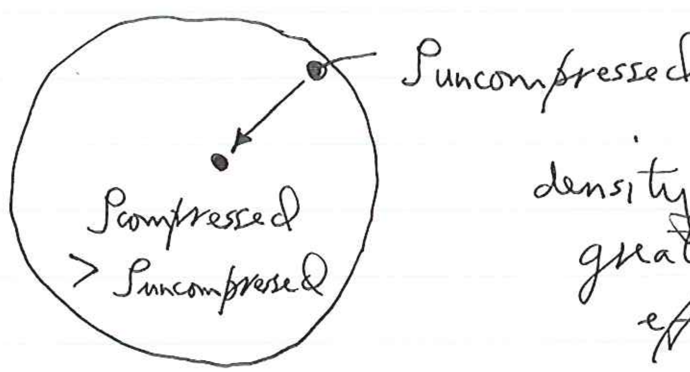
Our first hint regarding the composition of the $\oplus$: what is the $\oplus$ made of?

Not of common rocks collected at $\oplus$'s surface — basalts, granites, sedimentary rocks, etc.

Two reasons for discrepancy

First, the effect of compression

If the $\oplus$ were made out of 100% basalt, it would not have a mean density equal to that of basalt at 1 atm.



density would be greater due to effect of compression

We now know enough about density profile $\rho_{\oplus}(r)$ to correct for this compression effect.

$$\boxed{\bar{\rho}_{uncompressed} = 4300\ \mathrm{kg/m^3}}$$

Even after this correction

$$\boxed{\underbrace{\bar{\rho}_{\oplus} > \rho_{\text{rocks (silicates)}}}_{\text{both uncompressed}}}$$

First evidence that the Earth has a much denser <u>iron</u> <u>core</u>.

Is there any other constraint upon the Earth's (or other planets') internal density distribution that can be directly measured ?

Yes — the <u>moment of inertia</u> — a measure of how concentrated the mass is toward the center

$$\boxed{C = \frac{8\pi}{3} \int_0^R \rho(r)\, r^4\, dr}$$

Mass shell
basketball

mass M

$C = MR^2$

Uniform sphere

$C = \frac{2}{5} MR^2 = 0.4\, MR^2$

Mass more ~~[illegible]~~ concentrated in center

$$C < 0.4\, MR^2$$

The Earth's moment of inertia can be measured by measuring its oblateness or flattening or ellipticity

The argument goes back to Newton (Principia 1687)

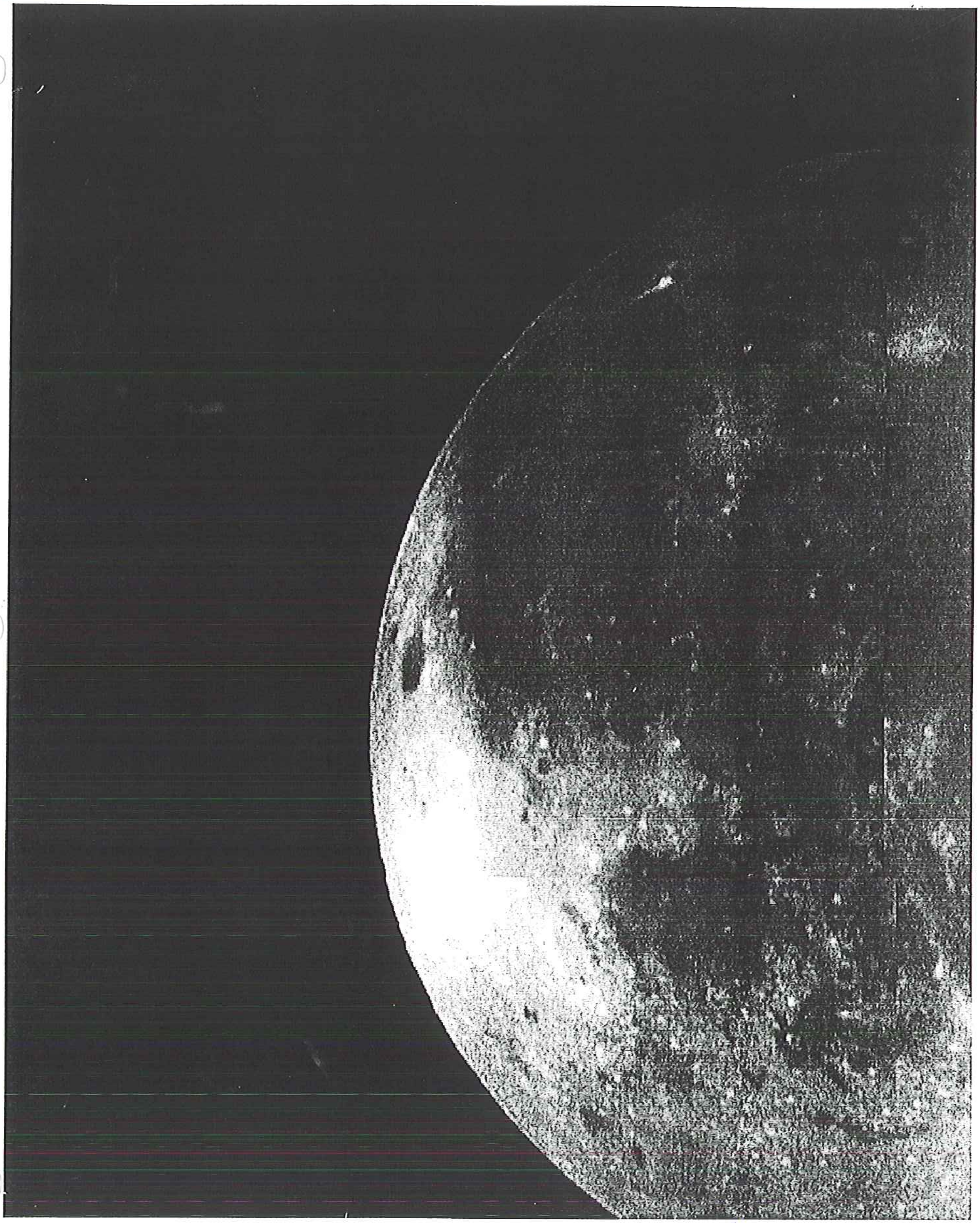
He pointed out that the Earth should be a flattened rather than a perfect sphere — an oblate ellipsoid — by virtue of its rotation.

Background — a sphere is the ideal natural shape of an isolated body of fluid.

Large, slowly rotating planetary bodies (e.g. the moon — rotation rate = orbital period = 1 month)

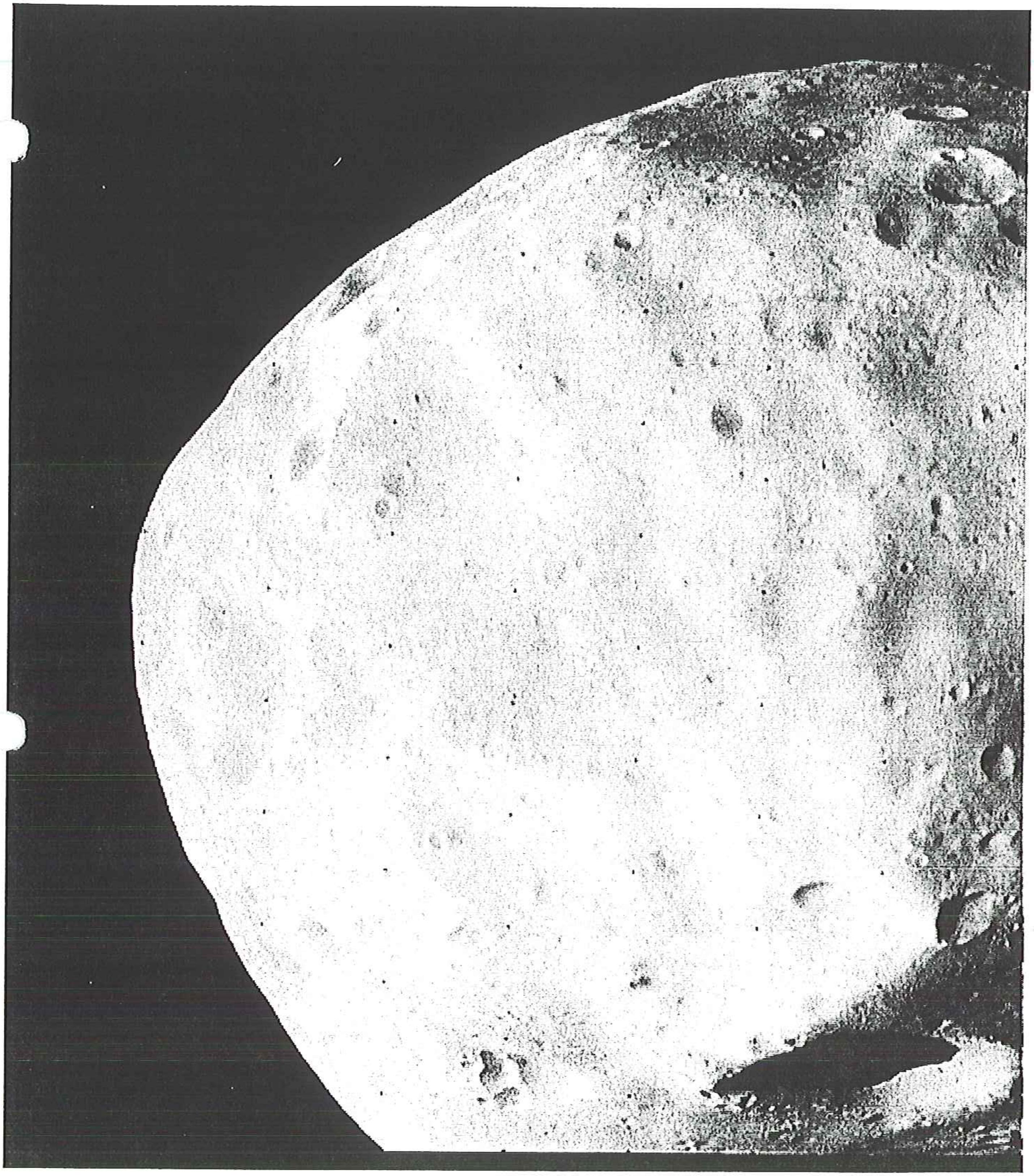
Smaller bodies, e.g. Phobos, need not be spherical.

On large scales and over long times the Earth and other planets behave



Moon 1740 km

if Moon were shrunk to size of a billiard ball, it would be very smooth

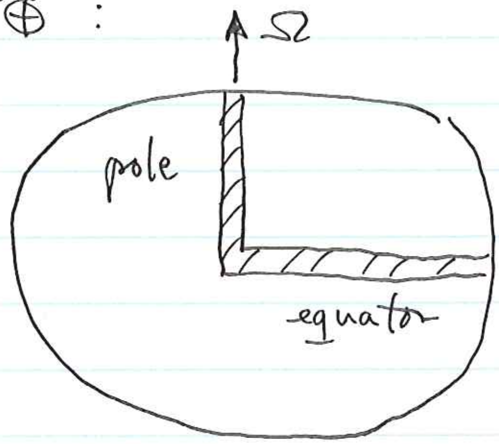


Phobos 16 km x 24 km

like fluids — weak.

Newton modelled the $\oplus$ as a homogeneous ($\rho =$ constant) fluid

Imagined drilling 2 holes to center of $\oplus$:



water in polar column attracted downward by gravity

ditto in equatorial column — but it is also thrown outward by centrifugal force

$\Omega =$ rotation rate $= \frac{2\pi}{86141 \text{ secs}} = 7.292 \cdot 10^{-5}$ rad/sec

Cause of flattening α

$$m = \frac{\text{centrifugal force}}{\text{gravity}} = \frac{\Omega^2 R}{GM/R^2} = \frac{\Omega^2 R^3}{GM}$$

Measure of effect — flattening or ellipticity

$$\varepsilon = \frac{R_{eq} - R_{pole}}{R_{eq}} \quad \text{or} \quad 1 - \varepsilon = \frac{R_{pole}}{R_{eq}}$$

In a homogeneous sphere, gravity decreases linearly with depth (Newton had to invent the integral calculus to prove this!) so the <u>dilution factor</u> m is constant

not really true — in fact his proof uses only Euclidean geometry!

weight of eq. column $= \frac{1}{2} R_{eq}\, g_{eq}\, (1-m)$ ← dilution factor

since $g(r) \propto r$

at ~~[illegible]~~ equator <u>on</u> <u>surface</u>

NB — it's a 1×1 m^2 column

weight of polar column

$= \frac{1}{2} R_{pole}\, g_{pole}$ no dilution

give this first

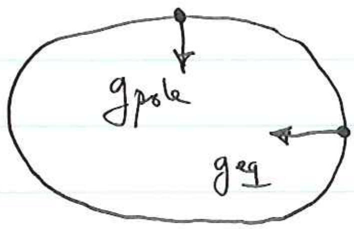
Equate:

$$\tfrac{1}{2} R_{pole}\, g_{pole} = \tfrac{1}{2} R_{eq}\, g_{eq}\, (1-m)$$

$$1-m = \frac{R_{pole}}{R_{eq}} \frac{g_{pole}}{g_{eq}}$$

$$= (1-\varepsilon) \frac{g_{pole}}{g_{eq}}$$

Newton could also solve for the second factor



$g_{pole} > g_{eq}$ since closer to $COM_{\oplus}$

In fact,

$$\boxed{\frac{g_{pole}}{g_{eq}} = 1 + \frac{1}{5}\varepsilon}$$

$$1 - m = (1-\varepsilon)\left(1 + \frac{1}{5}\varepsilon\right)$$

$$= 1 - \frac{4}{5}\varepsilon \cancel{- \frac{1}{5}\varepsilon^2} \quad \text{neglect } \varepsilon^2$$

neglect

$$m = \frac{4}{5}\varepsilon \quad \text{or} \quad \boxed{\varepsilon = \frac{5}{4} m}$$

$$m = \frac{\Omega^2 R^3}{GM} \qquad \Omega = \frac{2\pi}{1\ \text{day}} = \frac{2\pi}{86141\ \text{sec}}$$

one sidereal day

$$\Omega = 7.292 \cdot 10^{-5}\ \text{rad/sec}$$

$$\boxed{m = 1/290 \Rightarrow \varepsilon_{Newton} \simeq 1/230}$$

Newton's result valid only for a homogeneous sphere : $\rho(r) = $ constant

More general result obtained ~ 200 years later by Clairaut & Radau

$$\boxed{\varepsilon = \frac{10\, m}{4 + 25\left(1 - \frac{3}{2}\frac{C}{MR^2}\right)^2}}$$

Expresses flattening ε in terms of $m = \Omega^2 R^3 / GM$ (known) and C/MR^2

Thus by measuring ε one can determine C/MR^2.

Check — homogeneous sphere (Newton's case)
$C/MR^2 = \frac{2}{5}$

$$\varepsilon = \frac{10m}{4 + 25\left(1 - \frac{3}{2}\frac{2}{5}\right)^2} = \frac{5}{4} m \text{ — check}$$

Read underlined quotes from Principia to convey flavor of Newton's archaic language.

He predicted that $\oplus$ should have an equatorial bulge

$$\frac{6371 \text{ km}}{230} = 28 \text{ km} = \text{"}17\tfrac{1}{10} \text{ English miles"}$$

PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA.

Autore *J S. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

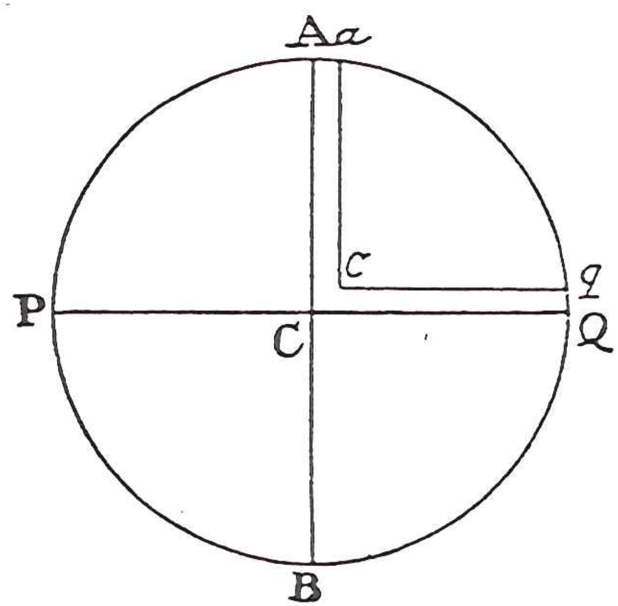
IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.

TITLE PAGE OF THE FIRST EDITION OF THE PRINCIPIA

Verum vis centrifuga partis cujusque est ad pondus ejusdem ut 1 et 289, hoc est, vis centrifuga, quæ deberet esse ponderis pars $\frac{4}{505}$, est tantum pars $\frac{1}{289}$. Et propterea dico, secundum Regulam auream, quod si vis centrifuga $\frac{4}{505}$ faciat ut altitudo aquæ in crure $ACca$ superet altitudinem aquæ in crure $QCcq$ parte centesimâ totius altitudinis: vis centrifuga $\frac{1}{289}$ faciet ut excessus altitudinis in crure $ACca$ sit altitudinis in crure altero $QCcq$ pars tantum $\frac{1}{229}$.



occasioned by some spots in that part of its body, which is then turned towards the earth, as M. *Cassini* has observed. So also the utmost satellite of Jupiter seems to revolve about its axis with a like motion, because in that part of its body which is turned from Jupiter it has a spot, which always appears as if it were in Jupiter's own body, whenever the satellite passes between Jupiter and our eye.

PROPOSITION XVIII. THEOREM XVI

That the axes of the planets are less than the diameters drawn perpendicular to the axes.

The equal gravitation of the parts on all sides would give a spherical figure to the planets, if it was not for their diurnal revolution in a circle. By that circular motion it comes to pass that the parts receding from the axis endeavor to ascend about the equator; and therefore if the matter is in a fluid state, by its ascent towards the equator it will enlarge the diameters there, and by its descent towards the poles it will shorten the axis. So the diameter of Jupiter (by the concurring observations of astronomers) is found shorter between pole and pole than from east to west. And, by the same argument, if our earth was not higher about the equator than at the poles, the seas would subside about the poles, and, rising towards the equator, would lay all things there under water.

PROPOSITION XIX. PROBLEM III

To find the proportion of the axis of a planet to the diameters perpendicular thereto.

Our countryman, Mr. *Norwood*, measuring a distance of 905751 feet of *London* measure between *London* and *York*, in 1635, and observing the difference of latitudes to be 2° 28′, determined the measure of one degree to be 367196 feet of *London* measure, that is, 57300 *Paris* toises. M. *Picard*, measuring an arc of one degree, and 22′ 55″ of the meridian between *Amiens* and *Malvoisine*, found an arc of one degree to be 57060 *Paris* toises. M. *Cassini*, the father, measured the distance upon the meridian from the town of *Collioure* in *Roussillon* to the Observatory of *Paris*; and his son added the distance from the Observatory to the Citadel of *Dunkirk*. The whole distance was 486156½ toises and the difference of the latitudes of *Collioure* and *Dunkirk* was 8 degrees, and 31′ 11⅚″. Hence an arc of one

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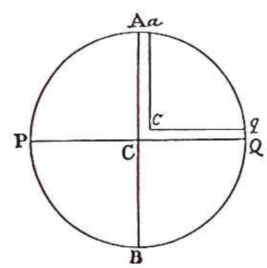
degree appears to be 57061 *Paris* toises. And from these measures we conclude that the circumference of the earth is 123249600, and its semidiameter 19615800 *Paris* feet, upon the supposition that the earth is of a spherical figure.

In the latitude of *Paris* a heavy body falling in a second of time describes 15 *Paris* feet, 1 inch, 1⅞ lines, as above, that is, 2173⅞ lines. The weight of the body is diminished by the weight of the ambient air. Let us suppose the weight lost thereby to be $\frac{1}{11000}$ part of the whole weight; then that heavy body falling in a vacuum will describe a height of 2174 lines in one second of time.

A body in every sidereal day of $23^{h}.\ 56^{m}.\ 4^{s}.$ uniformly revolving in a circle at the distance of 19615800 feet from the centre, in one second of time describes an arc of 1433.46 feet; the versed sine of which is 0.05236561 feet, or 7.54064 lines. And therefore the force with which bodies descend in the latitude of *Paris* is to the centrifugal force of bodies in the equator arising from the diurnal motion of the earth as 2174 to 7.54064.

The centrifugal force of bodies in the equator is to the centrifugal force with which bodies recede directly from the earth in the latitude of *Paris,* 48° 50′ 10″, as the square of the ratio of the radius to the cosine of the latitude, that is, as 7.54064 to 3.267. Add this force to the force with which bodies descend by their weight in the latitude of *Paris,* and a body, in the latitude of *Paris,* falling by its whole undiminished force of gravity, in the time of one second, will describe 2177.267 lines, or 15 *Paris* feet, 1 inch, and 5.267 lines. And the total force of gravity in that latitude will be to the centrifugal force of bodies in the equator of the earth as 2177.267 to 7.54064, or as 289 to 1.

Therefore if APBQ represent the figure of the earth, now no longer spherical, but generated by the rotation of an ellipse about its lesser axis PQ; and ACQ*qca* a canal full of water, reaching from the pole Q*q* to the centre C*c*, and thence rising to the equator A*a*; the weight of the water in the leg of the canal AC*ca* will be to the weight of water in the other leg QC*cq* as 289 to 288, because the centrifugal force arising from the



PQ: rotation axis

circular motion sustains and takes off one of the 289 parts of the weight (in the one leg), and the weight of 288 in the other sustains the rest. But by computation (from Cor. II, Prop XCI, Book I) I find, that, if the matter of the earth was all uniform, and without any motion, and its axis PQ were to the diameter AB as 100 to 101, the force of gravity in the place Q towards the earth would be to the force of gravity in the same place Q towards a sphere described about the centre C with the radius PC, or QC, as 126 to 125. And, by the same argument, the force of gravity in the place A towards the spheroid generated by the rotation of the ellipse APBQ about the axis AB is to the force of gravity in the same place A, towards the sphere described about the centre C with the radius AC, as 125 to 126. But the force of gravity in the place A towards the earth is a mean proportional between the forces of gravity towards the spheroid and this sphere; because the sphere, by having its diameter PQ diminished in the proportion of 101 to 100, is transformed into the figure of the earth; and this figure, by having a third diameter perpendicular to the two diameters AB and PQ diminished in the same proportion, is converted into the said spheroid; and the force of gravity in A, in either case, is diminished nearly in the same proportion. Therefore the force of gravity in A towards the sphere described about the centre C with the radius AC, is to the force of gravity in A towards the earth as 126 is to 125½. And the force of gravity in the place Q towards the sphere described about the centre C with the radius QC, is to the force of gravity in the place A towards the sphere described about the centre C with the radius AC, in the proportion of the diameters (by Prop. LXXII, Book I), that is, as 100 to 101. If, therefore, we compound those three proportions 126 to 125, 126 to 125½, and 100 to 101, into one, the force of gravity in the place Q towards the earth will be to the force of gravity in the place A towards the earth as 126 · 126 · 100 to 125 · 125½ · 101; or as 501 to 500.

Now since (by Cor. III, Prop. XCI, Book I) the force of gravity in either leg of the canal ACca, or QCcq, is as the distance of the places from the centre of the earth, if those legs are conceived to be divided by transverse, parallel, and equidistant surfaces, into parts proportional to the wholes, the weights of any number of parts in the one leg ACca will be to the weights of the same number of parts in the other leg as their magnitudes and the accelerative forces of their gravity conjointly, that is, as 101 to 100, and 500

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› conjointly, that is, as 101 to 100, and 500

to 501, or as 505 to 501. And therefore if the centrifugal force of every part in the leg AC*ca,* arising from the diurnal motion, was to the weight of the same part as 4 to 505, so that from the weight of every part, conceived to be divided into 505 parts, the centrifugal force might take off four of those parts, the weights would remain equal in each leg, and therefore the fluid would rest in an equilibrium. But the centrifugal force of every part is to the weight of the same part as 1 to 289; that is, the centrifugal force, which should be $\frac{4}{505}$ parts of the weight, is only $\frac{1}{289}$ part thereof. And, therefore, I say, by the rule of proportion, that if the centrifugal force $\frac{4}{505}$ make the height of the water in the leg AC*ca* to exceed the height of the water in the leg QC*cq* by $\frac{1}{100}$ part of its whole height, the centrifugal force $\frac{1}{289}$ will make the excess of the height in the leg AC*ca* only $\frac{1}{289}$ part of the height of the water in the other leg QC*cq*; and therefore the diameter of the earth at the equator[1] is to its diameter from pole to pole as 230 to 229. And since the mean semidiameter of the earth, according to *Picard's* mensuration, is 19615800 *Paris* feet, or 3923.16 miles (reckoning 5000 feet to a mile), the earth will be higher at the equator than at the poles by 85472 feet, or 17 1/10 miles. And its height at the equator will be about 19658600 feet, and at the poles 19573000 feet.

dilution
factor
constant

If, the density and periodic time of the diurnal revolution remaining the same, the planet was greater or less than the earth, the proportion of the centrifugal force to that of gravity, and therefore also of the diameter between the poles to the diameter at the equator, would likewise remain the same. But if the diurnal motion was accelerated or retarded in any proportion, the centrifugal force would be augmented or diminished nearly in the same proportion squared; and therefore the difference of the diameters will be increased or diminished in the same squared ratio, very nearly. And if the density of the planet was augmented or diminished in any proportion, the force of gravity tending towards it would also be augmented or diminished in the same proportion: and the difference of the diameters on the contrary would be diminished in proportion as the force of gravity is augmented, and augmented in proportion as the force of gravity is diminished. Therefore, since the earth, in respect of the fixed stars, revolves in $23^h. 56^m.$, but Jupiter in $9^h. 56^m.$, and the squares of their periodic times are as 29 to 5, and their densities as 400 to 94½, the difference of the diameters of Jupiter

[1 Appendix, Note 41.]

ELLIPSOIDAL FIGURES
OF EQUILIBRIUM

by S. Chandrasekhar

New Haven and London, Yale University Press, 1969

1

HISTORICAL INTRODUCTION

1. Newton

THE study of the gravitational equilibrium of homogeneous uniformly rotating masses began with Newton's investigation on the figure of the earth (*Principia*, Book III, Propositions XVIII–XX). Newton showed that the effect of a small rotation on the figure *must* be in the direction of making it slightly oblate; and, further, that the equilibrium of the body will demand a simple proportionality between the *effect* of rotation, as measured by the ellipticity,

$$\epsilon = \frac{\text{equatorial radius}-\text{polar radius}}{\text{the mean radius } (R)}, \tag{1}$$

and its *cause*, as measured by

$$m = \frac{\text{centrifugal acceleration at the equator}}{\text{mean gravitational acceleration on the surface}}$$
$$= \frac{\Omega^2 R}{GM/R^2} = \frac{\Omega^2 R^3}{GM}, \tag{2}$$

where G denotes the constant of gravitation and M is the mass of the body. More precisely, Newton established the relation

$$\epsilon = \tfrac{5}{4}m \tag{3}$$

in case the body is homogeneous. The arguments by which Newton derived this relation are magisterial; and they are worth recalling.

Newton imagined a hole of unit cross-section drilled from a point on the equator to the center of the earth and a similar hole drilled from the pole to the center; and he further imagined that the "canals" so constructed were filled with a fluid (see Fig. 1, after Newton's original illustration in the *Principia*). From the fact that the fluid in the canals will be in equilibrium, Newton concludes that the "weights" of the equatorial and the polar columns of the fluid must be equal. However, along the equator the acceleration due to gravity is "diluted" by the centrifugal acceleration; and since both these accelerations in a homogeneous body vary from the center proportionately with the distance, the "dilution factor" remains constant and is given by its value at the boundary, namely m.

If a denotes the equatorial radius, the weight of the equatorial column is given by

$$\text{weight of equatorial column} = \tfrac{1}{2}a g_{\text{equator}}(1-m), \qquad (4)$$

where g_{equator} is the acceleration due to gravity at the equator. Similarly, if b denotes the polar radius,

$$\text{weight of polar column} = \tfrac{1}{2}b g_{\text{pole}}. \qquad (5)$$

And since the two weights must be equal,

$$a g_{\text{equator}}(1-m) = b g_{\text{pole}}. \qquad (6)$$

But for a slightly oblate body Newton knew that

$$\frac{g_{\text{pole}}}{g_{\text{equator}}} = 1+\tfrac{1}{5}\epsilon+O(\epsilon^2). \qquad (7)$$

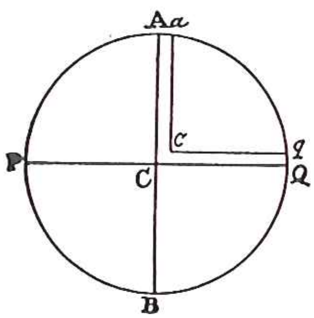

FIG. 1. Illustration from the *Principia* bearing on Newton's arguments for the rotational flattening of the earth.

Equations (6) and (7) and the definition of ϵ $(= 1-b/a)$ now give

$$1-m = (1-\epsilon)(1+\tfrac{1}{5}\epsilon)+O(\epsilon^2) = 1-\tfrac{4}{5}\epsilon+O(\epsilon^2); \qquad (8)$$

and Newton's relation (3) follows.

It was known already in Newton's time that

$$m = \tfrac{1}{290}. \qquad (9)$$

Therefore, Newton concluded that if the earth were homogeneous, it should be oblate with an ellipticity

$$\epsilon = \tfrac{5}{4}\tfrac{1}{290} \simeq \tfrac{1}{230}. \qquad (10)$$

This prediction of Newton was contrary to the astronomical evidence of the time and "two generations of the best astronomical observers formed in the school of the Cassinis struggled in vain against the authority and reasoning of Newton" (Todhunter's *History*, **1**, 100). The opposing ideas of Newton and Cassini are strikingly illustrated in the accompanying old caricature (Fig. 2). However, geodetic measurements made in Lapland by Maupertuis and Clairaut (1738) afforded data which conclusively showed the flattening of the earth at the poles. As Todhunter has written (**1**, 100), "The success of the arctic expedition may be ascribed in great measure to the skill and energy of Maupertuis; and his fame was widely celebrated. The engravings of the period represent him in the costume of a Lapland Hercules having a fur cap over his eyes; with one hand he holds a club and with the other he compresses the

terrestrial globe." And
him warmly for having "
and Voltaire became inv
wrote Vouz av
Ce que N

FIG. 2. An
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We know now that t
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2. Maclaurin

The next advance (17
generalized Newton's re
the rotation cannot be c

Maclaurin had solved e
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acceleration due to gravit

$g_{\text{equator}} =$

and $g_{\text{pole}} =$

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eccentricity. And since b
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g_{equ}

or $\Omega^2 =$

terrestrial globe." And Voltaire, then Maupertuis' friend, congratulated him warmly for having "aplati les poles et les Cassini." Later Maupertuis and Voltaire became involved in a heroic-comic controversy and Voltaire wrote

> Vouz avez confirmé dans les lieux pleins d'ennui
> Ce que Newton connut sans sortir de chez lui.

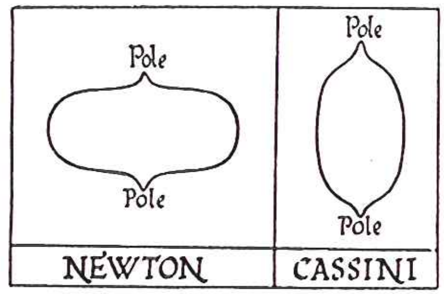


FIG. 2. An old-time caricature of the controversy between the opposing schools of Newton and Cassini with respect to the figure of the earth.

We know now that the actual ellipticity of the earth ($\sim 1/294$) is substantially smaller than Newton's predicted value ($\sim 1/230$); and this discrepancy is interpreted in terms of the inhomogeneity of the earth.

2. Maclaurin

The next advance (1742) in the theory was due to Maclaurin who generalized Newton's result to the case when the ellipticity caused by the rotation cannot be considered small.

Maclaurin had solved earlier the problem of the attraction of an oblate spheroid at an internal point; and he had shown in particular that the acceleration due to gravity at the equator and at the poles have the values

$$g_{\text{equator}} = 2\pi G\rho a\frac{(1-e^2)^{\frac{1}{2}}}{e^3}[\sin^{-1}e-e(1-e^2)^{\frac{1}{2}}]$$

and

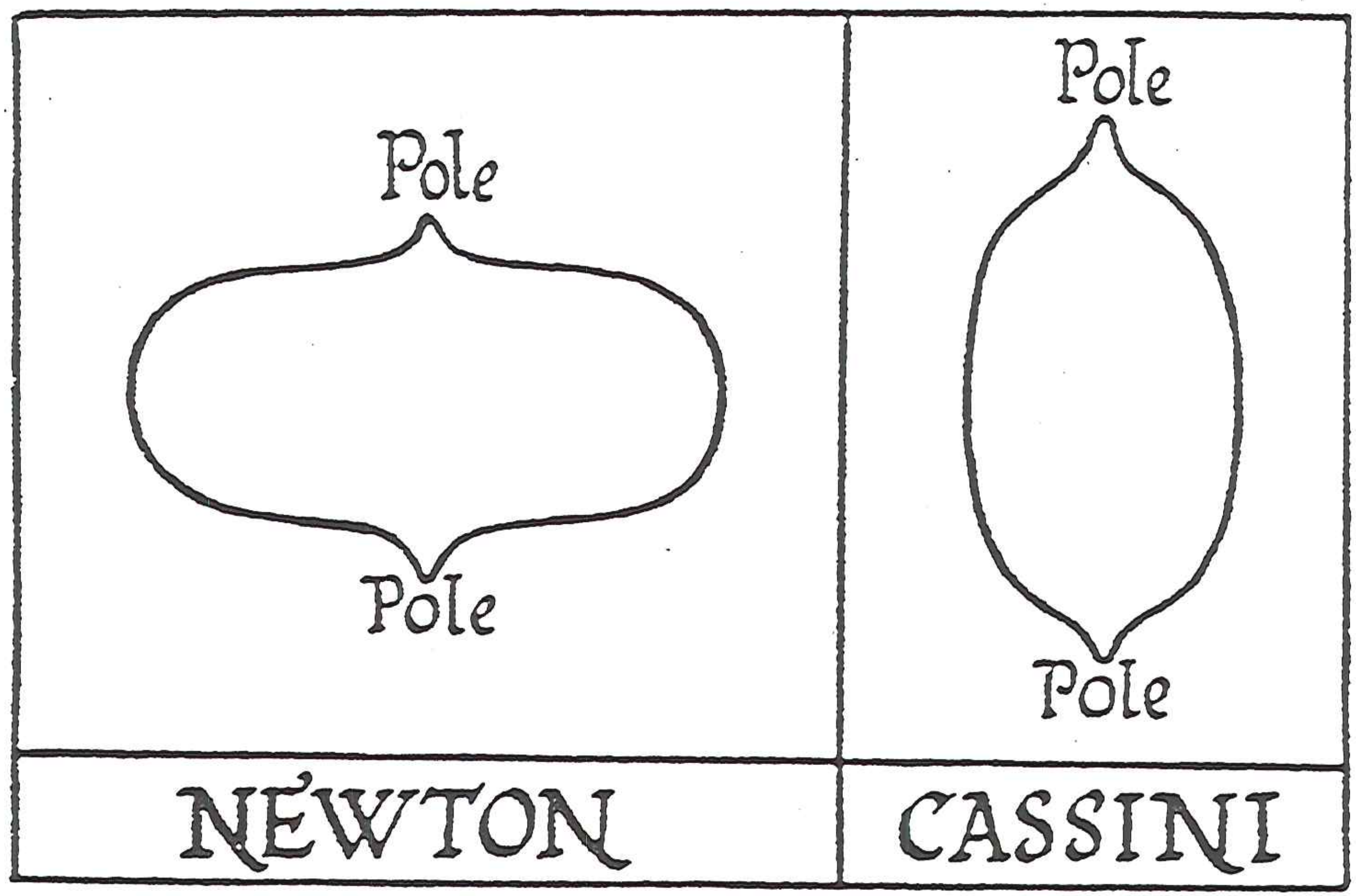
$$g_{\text{pole}} = 4\pi G\rho a\frac{(1-e^2)^{\frac{1}{2}}}{e^3}[e-(1-e^2)^{\frac{1}{2}}\sin^{-1}e], \tag{11}$$

where ρ is the density of the spheroid, a its semi-major axis, and e its eccentricity. And since both the centrifugal acceleration in the equatorial plane and the acceleration due to gravity vary linearly with the coordinates, Newton's argument applies to this case equally well and we can write

$$g_{\text{equator}}-a\Omega^2 = g_{\text{pole}}(1-e^2)^{\frac{1}{2}},$$

or

$$\Omega^2 = \frac{1}{a}[g_{\text{equator}}-g_{\text{pole}}(1-e^2)^{\frac{1}{2}}]. \tag{12}$$



Pole
Pole
NEWTON
Pole
Pole
CASSINI

In conflict with then best astronomical determinations

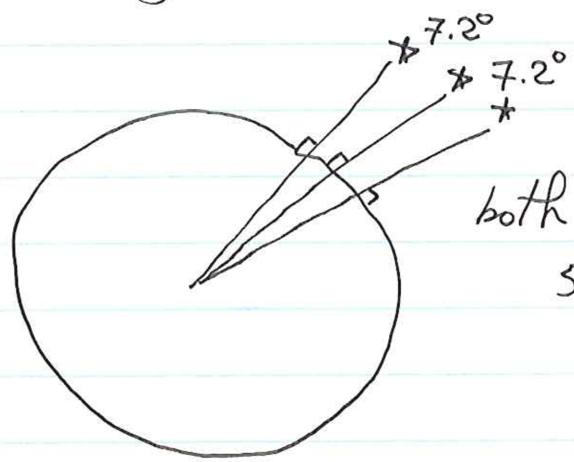
J.D. & Jacques Cassini — father & son — Observatoire de Paris

Their measurements showed $\oplus$ to be flattened at equator, not at poles!

How does one measure the $\oplus$'s shape?

By surveying — precise measurements of angles and distances on the ground.

Background — if the $\oplus$ were perfectly spherical

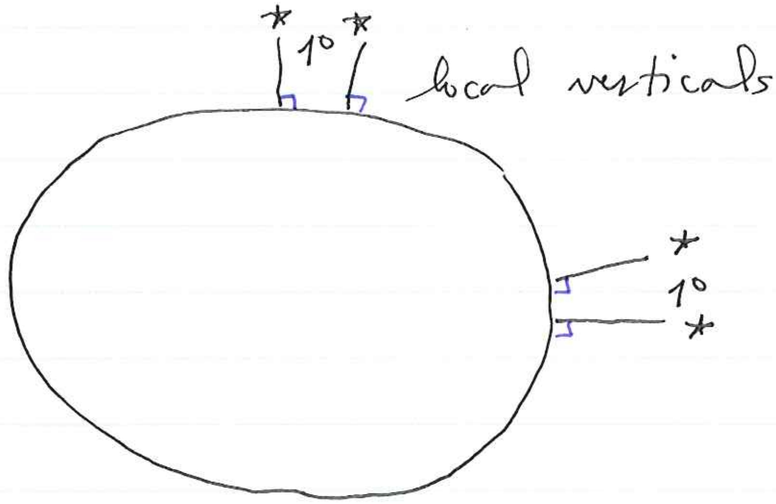


both distances 5000 stadia = 800 km

i.e. equal angles (between ✶) would subtend equal-length arcs on the surface of the $\oplus$.

On the other hand, on a Newtonian flattened $\oplus$:

Arclength of 1° of latitude should <u>increase</u> toward the poles



Cavaillon to Dunkirk

Cassinis surveyed a meridian in France.
Toujours chauvinistes, les Français

Measurements not accurate enough — concluded ⊕ was football or lemon rather than pumpkin-shaped.

Led to considerable argument — eventually became a major issue of French / British pride — satirized by Swift in <u>Gulliver's Travels</u>

Not resolved until 1730's

1734 Louis <u>XV</u> & French Academy of Sciences commissioned two expeditions, to Lapland and Peru. First to return was Maupertuis in 1736

Two summers in tundra — triangulated between mountain tops (which had to be cleared) to measure distances — used a <u>toise</u> or rod of fir to measure the <u>baseline</u>

almost lost toise in a shipwreck on way home

Found arclength of 1° latitude in Lapland = 110.69 km (at 65° N) compared to 110.46 km in France (at 45° N)

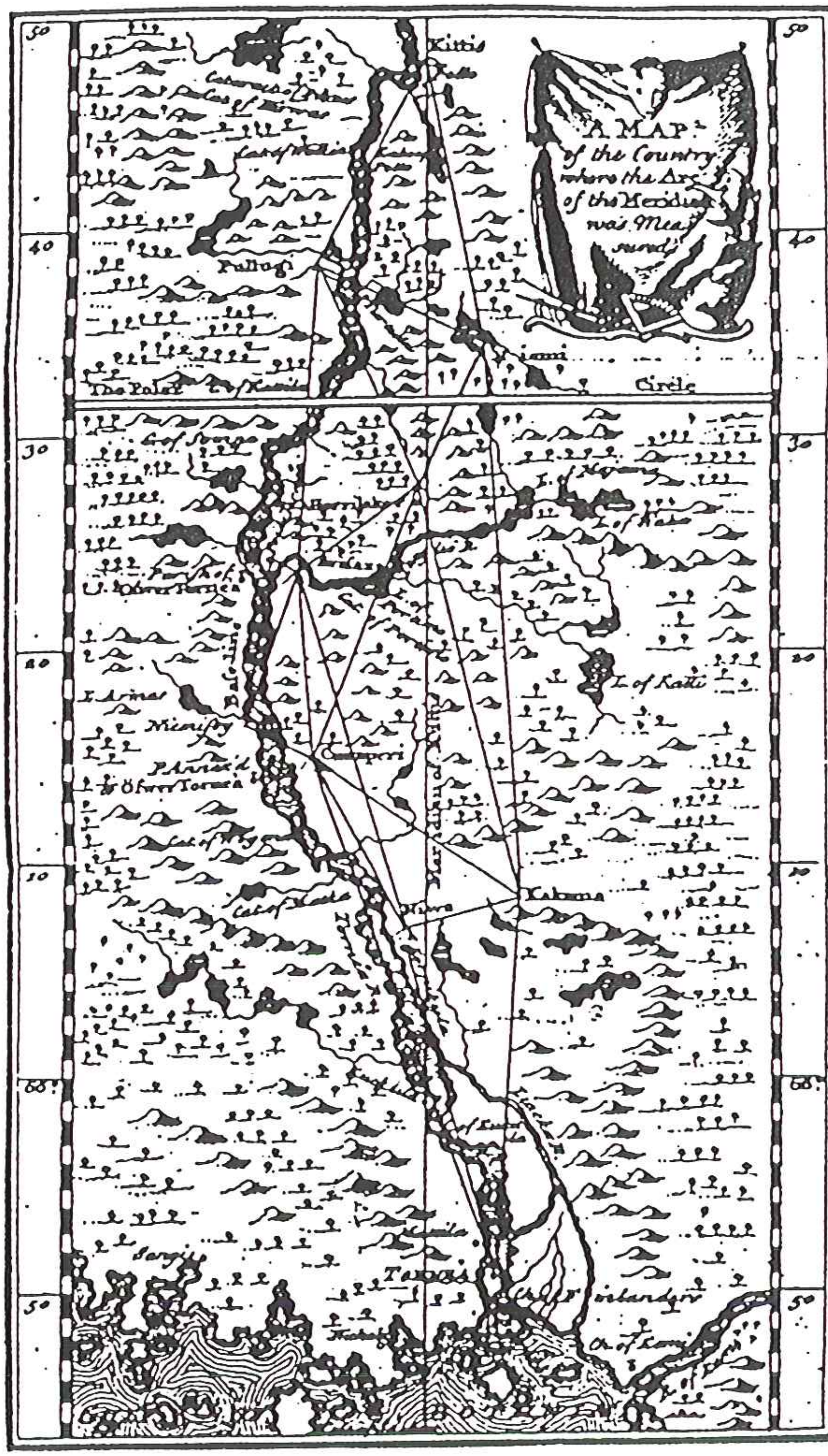
Voltaire — "vous avez aplati les pôles et les Cassinis"

Later, he chided Maupertuis in a mock-heroic poem

"Vous avez confirmé dans ces lieux pleins d'ennui ce que Newton connu sans sortir de chez lui"

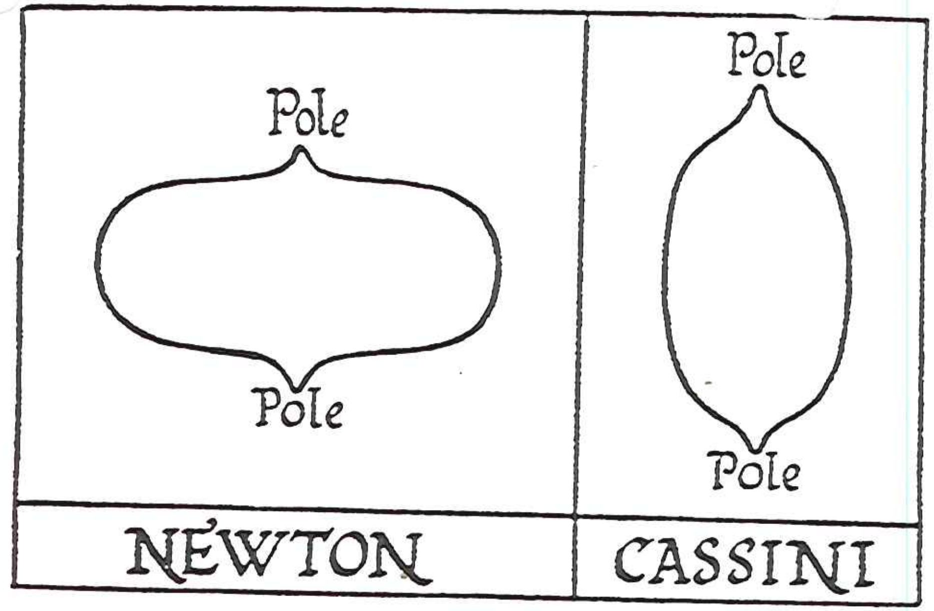
Rival expedition to Peru led by Bouguer — did not return until 1740

Two rival expeditions found two values of flattening 1/179 and 1/266 — mean = 1/230 !



A map of Lapland compiled by the French scientist Maupertuis in 1736–37, to show his triangulation of a degree of a meridian. He began at Tornio (Torneå) at 65°50′N and stopped at Kittis Mountain just short of 66°50′N and just north of the Arctic Circle. His calculation, when compared with the length of a degree in France and on the equator in Peru, showed that the Earth was slightly flattened at the poles, as Newton's theory of gravitation had predicted.

Les fils de Cassini, héritiers des traditions paternelles, niaient les principales conséquences que Newton avait tirées de la gravité universelle, l'aplatissement des pôles par exemple. Non-seulement les Cassini contestaient cet aplatissement, mais ils prétendaient que la terre était un sphéroïde allongé dans le sens des pôles, et les faits semblaient leur donner raison. La géométrie montre que dans un sphéroïde aplati la longueur des degrés va en augmentant à mesure qu'on avance de l'équateur vers les pôles. On avait mesuré plusieurs arcs de méridien dans les premières années du siècle, on avait fait notamment une mesure en France, entre les Pyrénées et Dunkerque, et l'on trouvait que les degrés étaient d'autant plus petits qu'on approchait plus du nord; on en concluait naturellement qu'on avait affaire à un sphéroïde allongé. C'était là une circonstance d'un grand poids et qui tenait à elle seule en échec les partisans de Newton. Cependant la mesure du méridien faite en France inspirait des doutes. En 1735, l'Académie des Sciences organisa une grande expédition pour étudier cette question tant controversée. Bouguer et La Condamine partirent pour le Pérou. Clairaut et Maupertuis allèrent en Laponie, accompagnés de Camus et de Lemonnier comme assistans. En mesurant un arc près de l'équateur, un autre près du pôle, et comparant les résultats ainsi obtenus aux mesures exécutées en France, on devait avoir tous les élémens nécessaires pour trancher le litige. Cette vérification solennelle donna raison à ceux qui tenaient pour l'aplatissement du sphéroïde terrestre. Les travaux des quatre associés ne purent être réunis et comparés que vers 1740, la mission du Pérou ayant été retardée par divers contre-temps; mais dès l'année 1736 Maupertuis revint, rapportant les mesures prises en Laponie, et dont la comparaison avec les mesures françaises suffisait à la rigueur pour décider la question. Les degrés voisins du pôle étaient décidément les plus longs. Maupertuis proclama ce résultat, en fit retentir tous les échos; dès l'année 1738, sans attendre le retour de Bouguer et de La Condamine, il publia un livre sur *la Figure de la terre* qui fut considéré comme décisif. Il usurpa ainsi auprès du public la gloire de l'œuvre commune. Les gravures du temps le représentent, en costume de Lapon, écrasant de sa main le pôle du monde, et Voltaire, dont il était alors l'ami, put le féliciter hautement d'avoir « aplati les pôles et les Cassini. »



Certainement vous savez peindre; il ne tenait qu'à vous d'être nôtre plus grand poëte comme notre plus grand mathématicien. Si vos opérations sont d'Archimède, et votre courage de Christophe Colomb, votre description des neiges de Tornéo est de Michel-Ange, et celle des espéces d'aurores boréales est de l'Albane.

Voltaire, Lettre à Maupertuis après son retour.

Courriers de la physique, argonautes nouveaux,
Qui franchissez les monts, qui traversez les eaux,
Ramenez des climats soumis aux trois couronnes
Vos perches, vos secteurs, et sur-tout deux Lapones.
Vous avez confirmé dans ces lieux pleins d'ennui
Ce que Newton connut sans sortir de chez lui.

Voltaire, Discours sur la Modération

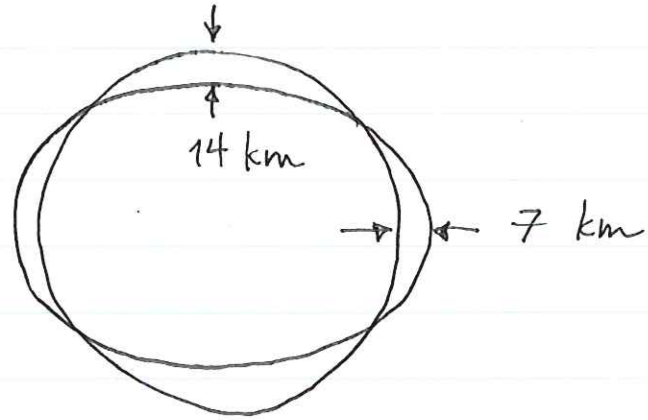
Modern more precise value — obtained by analysis of satellite orbits

$$\boxed{\varepsilon_{\substack{\text{best-fit}\\ \text{to geoid}}} = 1/298.25}$$

Inferred moment of inertia of $\oplus$:

$$\boxed{C/MR^2 = 0.33089 < 0.4}$$

So, to a very good approximation, the shape of the $\oplus$ is an ellipsoid of revolution with a flattening $\varepsilon = 1/298.25$



polar radius $R_{pole} = 6371 - 14 = 6357$

mean eq. radius $R_{eq} = 6371 + 7 = \underline{\underline{6378}}$ km

$R_{eq} - R_{pole} = \underline{\underline{21 \text{ km}}}$

Geoid = mean sea level (in absence of waves and tides)

On land — visualize as a network of canals

Denver — one mile above the geoid below

Geoid bumps — relative to $\varepsilon = 1/298.25$ ellipsoid

$+81$ m : New Guinea
-113 m : off S. tip of India

Surface of geoid is extremely smooth

Topography is much more rugged — but even so very smooth

Highest mountain (Everest) ~ 10 km high