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**tidal friction

Four lectures on tides : GFD 506.

These lectures will deal with both ocean tides and with Earth tides, since both are now known to be strongly influenced by the other. In the oceans, we shall restrict consideration to the barotropic tide or surface tide. Internal or baroclinic tides are not unimportant in oceanic dynamics, and are crucial for atmospheric tides, but they will not be discussed.

Because they are obvious and so important to seafaring nations, ocean tides have been observed far back into history. Before Newton, many wild explanations advanced.
Ch. IV of The Tides by G.H. Darwin (1898), charming Victorian book for a lay audience, p. 79 " displacement by foot of an angel, p. 76 " Chinese writers have suggested... that water is the blood of the Earth, and that the tides are the beating of its pulse."
Many early observers noticed correlation with ☾ motion and spring and neap tides with full/new and half ☾.

Early explanations: heat from ☉ drying up oceans, moisture from ☾.
Keenest early observers of natural phenomena (Greeks + Romans), on shores of Mediterranean, had little occasion to ponder over tides.

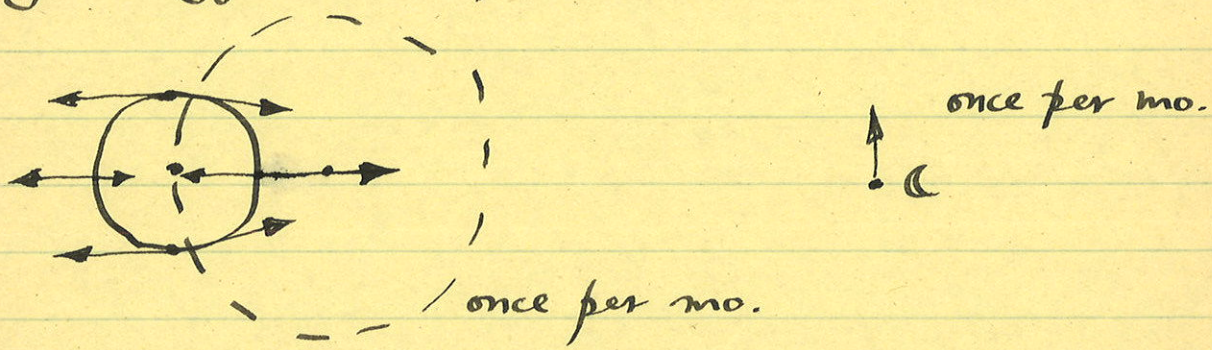
1687 Newton <u>Principia</u>: a correct explanation in terms of grav. attr. of ☾, ☉ on oceans.
His was a <u>static</u> theory, i.e. he assumed oceanic res<u>ponse</u> would be static or equilibrium
Led to concept of equi<u>librium</u> tide, still a useful concept.

Dominant tide is <u>semi-diurnal</u>, because dominant <u>equilibrium</u> tide is semi-diurnal.
Why? Neglect ☉, consider grav. attr. of ☾ at pts. on ⊕, varies from pt. to pt.

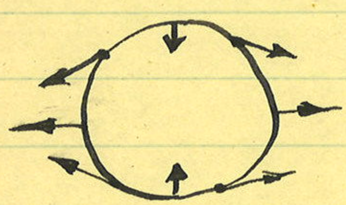
At first glance, should be one tide per lunar day as ⊕ rotates. But actually ⊕ and ☾ mutually orbit about their common c.o.m., actually located <u>in</u> ⊕.

Every pt. on ⊕ experiences the <u>same</u> <u>orbital</u> acceleration.

Highly exaggerated picture

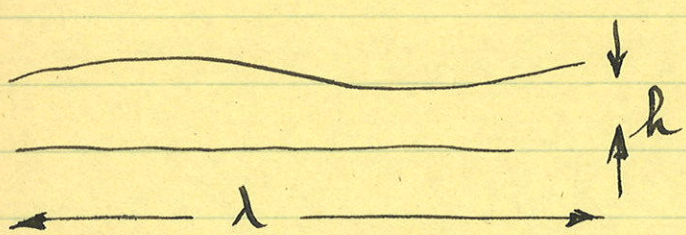


Resultant thus looks like.



If no ☉, ☾ equilibrium tide would be dominantly semi-diurnal (2 c per <u>lunar</u> day).

Next important development 1774 Laplace <u>Mechanique Celeste</u>: discussed ocean tides as a <u>dynamic</u> problem. Can think of ocean tide as a shallow water wave



$\lambda \gg h$, thus $c = \sqrt{gh}$, phase or group velocity

$h \sim 5$ km

$c = \sqrt{10^3 \cdot 5 \cdot 10^5} \sim 700$ km/hr

Free periods of major ocean basins ~ time to cross + return ~ 12 hrs. Thus there will be resonance effects; this invalidates a static theory.

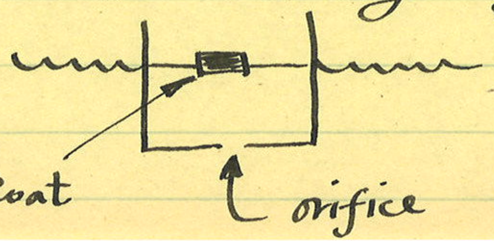
Furthermore, influence of Coriolis force on the normal modes of the oceans ~ Ω/σ, where σ ~ eigenfrequency. This is $O(1)$, rotation of ⊕ is an essential element in the response.

1774 Laplace derives LTE, the linearized eqns of motion governing dynamical response of oceans on a rigid underlying Earth (rotating) to a changing tidal potential. Numerical attacks on LTE have not been terribly successful.

Victorian era 1860's – 1900's was the heyday of tidal theory. Kelvin, George Darwin, Doodson, Proudman, Love (especially solid Earth tides)

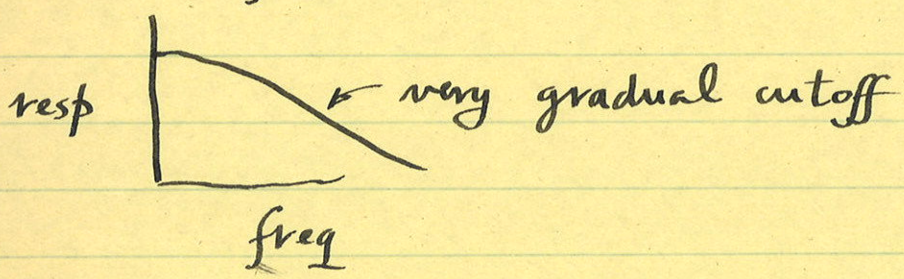
Lord Kelvin designed first self-recording tide gauge ~ 1860's. Previously used meter sticks (tide poles) read hourly by eye.

Kelvin tide gauge

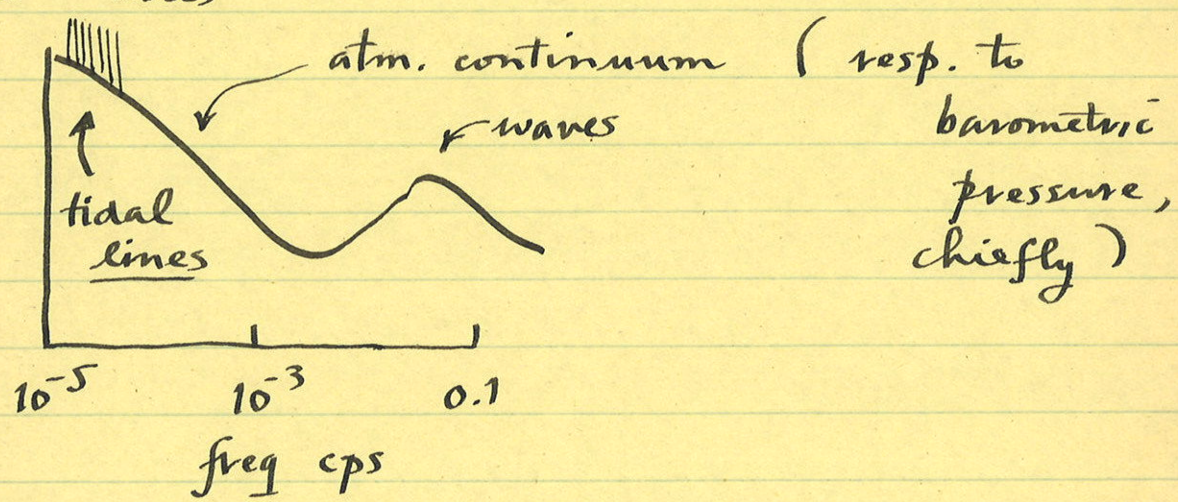


Float attached to a pencil or some other recording device. Simple mechanical filter to damp wave motion. Chief problems: siltation and "biological fouling". These gauges still in use in most ports today. Tidal records geophysically unique for their continuity.

The orifice works well in spite of

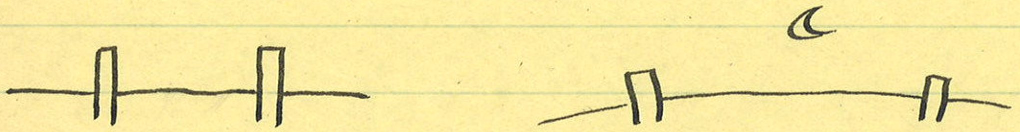


because of the minimum between waves and tides



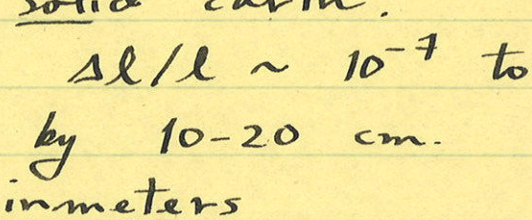
Study of ocean tides long hindered because ∃ only measurements on coasts and scattered islands. This is exactly where you would expect tides to be most complicated and maybe non-linear, because of purely local effects. Problem: how to measure tides in deep sea. Answer: pressure gauges on bottom; $p = \rho g h$, since tide (barotropic) is a shallow water wave. Modern technology: free fall retrievable capsules. If the purely technological problems can be overcome, should give the very best tidal records, high snr, very stable thermal environment, also noise more pronounced near coasts. Note: wave action is filtered by depth.

Kelvin 1876 quantitative discussion of elastic deformation of solid Earth. Produces tidal strains $\Delta l / l \sim 10^{-7}$ to 10^{-8}; surface of $\oplus$ elevated by 10-20 cm. Measured with strainmeters



$$\frac{\Delta l}{l} \sim 10^{-7} \text{ to } 10^{-8}$$

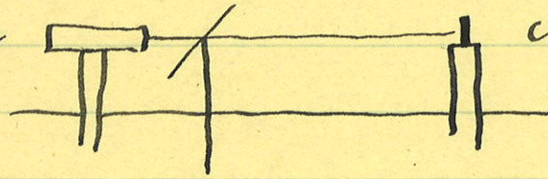
Benioff quartz (low α, still must site in a mine or tunnel). Typically uses a capacitive transducer



Measure $\omega = \sqrt{LC}$ of a tuned circuit. Typically $l \sim 50-100$ m. The longer the

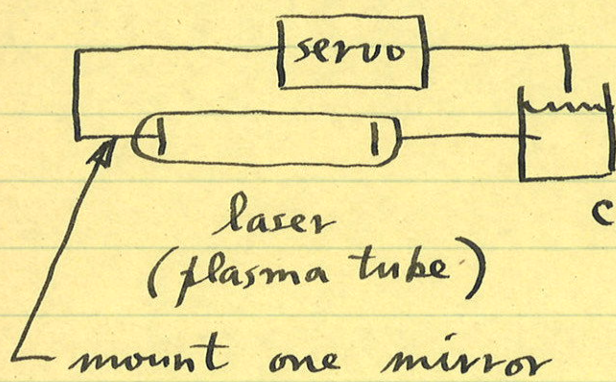
better (to average out geology effects).
If $l = 100$ m, $\Delta l \sim .01$ to $.001$ mm.

Laser interferometer, stable and precise, but rather \$K. Can count total # on one hand.



Michelson

Measures strain in fringes or λ of He-Ne light. Fringe splitting, since ~ 10 fringes ~ Δl
Two problems: 1. atmospheric refractive index variations $\gg$ tidal signal, must evacuate, expensive, pipe, pumps, esp for a 1 km. instrument. 2. laser mode jumping, must be stabilized, best method is methane absorption stabilization.



very narrow, stable absorption line

This raises # tag considerably.

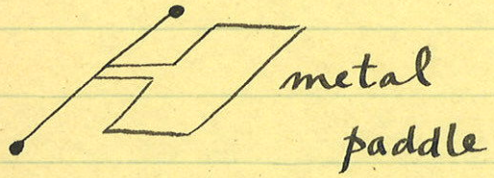
this is 0.1 mgal

$\exists$ also tidal gravity fluctuations $\Delta g/g \sim 10^{-7}$ to 10^{-8}

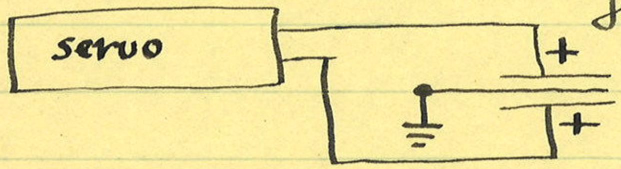
Measured with very long period seismometers (essentially a mass on a spring). Difficulty is securing a good drift-free long period response. Thermal control to millidegrees is essential

Lacoste-Romberg: secret ingredient, zero length spring.

Other type uses quartz (low α) torsion fibre

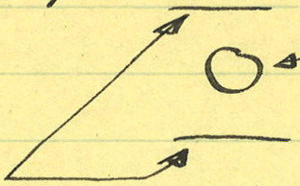


System is very linear: uses electrostatic capacitive feedback. Apply voltage V(t) to keep paddle centered.



Tricky to calibrate.

The ultimate ($\exists$ only one) is the superconducting gravimeter. Low thermal noise at cryogenic temps, stability controlled by inherent stability of superconducting currents (a quantum phenomenon)



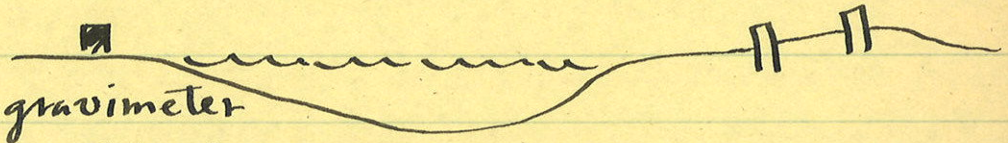
1" superconducting sphere levitated by eddy currents

capacitor plates for sensing posn and for servo linearization. As before, one measures V(t).

Main problem: refilling of liquid He dewar.

Earth tides are much more nearly static or equilibrium. Fund period ${}_0S_2 \sim 54$ min, or time for P wave to go thru ⊕ and return ~ 40 min. $\ll 12$ hr. Response will be quasi-static.

Main problem is contamination by ocean tidal loading



gravimeter sees direct attraction of ocean H_2O, gets lowered in ⊕ gravity field, sees effect of redistribution of ⊕ mass under load.

This is about 20% of total gravity tidal signal, smaller mid-continent, higher near coasts. In principle, Earth tides can put constraints on global ocean tides. Global ocean tides must be known if Earth tide data is to provide other interesting information. Example: what is the phase lag of the quasi-static ⊕ tide; can put a constraint on volume averaged anelasticity. At present, this hopelessly buried in 20% tidal loading signal.

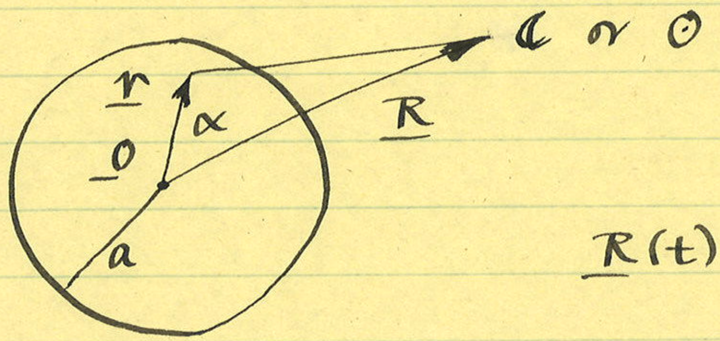
On the other hand, deformation of solid ⊕ has a quantitative effect on oceanic response

We shall defer until later the reason for this. Only recently appreciated, LTE (derived by Laplace for a <u>rigid</u> underlying Earth) <u>must</u> be modified to allow for yielding of solid $\oplus$.

<u>The tidal potential</u>: Assumptions ☾, $\odot$ are point masses, $\oplus$ is spherically symmetric, neglect $\varepsilon = 1/300$ ellipticity. Can be shown rigorously this can nowhere lead to errors larger than $O(\varepsilon)$.

Treat ☾, $\odot$ separately by superposition

We adopt a <u>geocentric</u> view.



$\underline{R}(t)$: posn vector of ☾, $\odot$.

Consider a <u>fixed</u> t.

Earth : radius a

$\underline{O}$: c.o.m. of $\oplus$

Pt. of "observation" not necessarily $\oplus$ surface

<u>Net</u> grav. force of ☾ on $\oplus$ is

$$\underline{f} = GM_{\oplus} M_{☾} \hat{\underline{R}} / R^2$$

Resulting <u>orbital acceleration</u> of $\oplus$ is

$$\underline{A} = \underline{f}/M_{\oplus} = GM_{☾} \hat{\underline{R}} / R^2$$

Note $\underline{A} \neq \underline{A}(\underline{r})$, constant throughout $\oplus$

Grav. pot. of ☾ at pt. $\underline{r}$ of observation

$$\phi_{☾}(\underline{r}) = -\frac{GM_{☾}}{r'} = -\frac{GM_{☾}}{|\underline{r}-\underline{R}|}$$

$$= -\frac{GM_{☾}}{R}\sum_{l=0}^{\infty}\left(\frac{r}{R}\right)^{l} P_l(\cos\alpha)$$

$$\cos\alpha = \hat{r}\cdot\hat{R}$$

We wish to view the tides <u>on</u> the <u>accelerating</u> $\oplus$. D'Alembert's principle: repr. acc. $\underline{A}$ by a fictitious force.

Net <u>tidal</u> force per unit mass on a test mass at $\underline{r}$, viewed in the accelerating frame is

$$-\nabla\phi_{☾}(\underline{r}) - \underline{A} = -\nabla\left[\phi_{☾}(\underline{r}) + \underline{r}\cdot\underline{A}\right]$$

Now consider $\phi_{☾}(\underline{r}) + \underline{r}\cdot\underline{A} =$

$$-\frac{GM_{☾}}{R}\left[1 + \frac{r}{R}\cos\alpha + \sum_{l=2}^{\infty}\left(\frac{r}{R}\right)^{l} P_l(\cos\alpha)\right]$$

$$+\frac{GM_{☾}}{R}\left(\frac{r}{R}\cos\alpha\right)$$

$$= -\underbrace{\frac{GM_{☾}}{R}}_{\text{constant}} - \frac{GM_{☾}}{R}\left[\sum_{l=2}^{\infty}\left(\frac{r}{R}\right)^{l} P_l(\cos\alpha)\right]$$

The constant = potential at $r = 0$, gives rise to no force. The $l = 1$ term has been cancelled by the orbital acceleration. This term gives rise to a uniform force throughout the ⊕ $GM_☾ \hat{R}/R^2$; this precisely the force keeps ⊕ accelerating in its Keplerian orbit. (One is in free fall while in orbit)

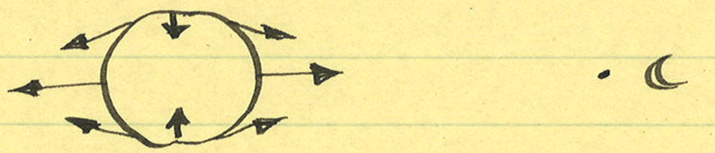
Tidal forces due to small variations throughout ⊕ of force / unit mass exerted by ☾.

$$\text{Let } V_☾(\underline{r}) = -\frac{GM_☾}{R}\left[\sum_{l=2}^{\infty}\left(\frac{r}{R}\right)^l P_l(\cos\alpha)\right]$$

This called lunar tidal potential

Tidal force $\underline{T}(\underline{r}) = -\nabla V(\underline{r})$

Pictorial repr. of $\underline{T}(a\hat{r})$ on surface of ⊕



Can be shown: let $\underline{f}_☾(\underline{r}) \equiv$ net force / unit mass at $\underline{r}$ exerted by ☾. Then $\underline{T}_☾(\underline{r}) = \underline{f}_☾(\underline{r}) - \underline{f}_☾(\underline{0})$. Tidal forces are due to small differences. The vector subtracted to give the resultant tidal forces above was exactly $\underline{f}_☾(\underline{0})$, the force at the origin.

Examine <u>relative</u> magnitude of terms in sum. At ⊕ surface $r = a$,

$$\left(\frac{a}{R}\right)^2 P_2(\cos\alpha) + \left(\frac{a}{R}\right)^3 P_3(\cos\alpha) + \cdots$$

For ☾ : $\frac{a}{R} \sim \frac{1}{60}$ rad (~ 1°, H.P. of ☾
☉ : $\frac{a}{R} \sim 1/23{,}000$ to a navigator)

For ☉, can certainly neglect all but $l=2$. For ☾, higher terms lead to equil. tides ~1/60 the ht. of $l=2$ tides. For many practical purposes, also can neglect, although these have been detected.

Thus $V_☾(\underline{r}) \approx -\frac{GM_☾}{R}\left(\frac{r}{R}\right)^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)$

Note $V_☾ \sim \frac{M_☾}{R_☾^3}$, $V_☉ \sim \frac{M_☉}{R_☉^3}$

What is relative effect of ☉, ☾ ?

$$☾/☉ \sim \frac{M_☾/R_☾^3}{M_☉/R_☉^3} = \frac{\bar{\rho}_☾ a_☾^3}{\bar{\rho}_☉ a_☉^3}\,\frac{R_☉^3}{R_☾^3}$$

$$= \frac{\bar{\rho}_☾}{\bar{\rho}_☉}\left(\frac{a_☾/R_☾}{a_☉/R_☉}\right)^3,$$ but $\frac{a}{R}$ = visible half aperture in sky

These ~ same (eclipses). Thus effect ~ ratio of $\bar{\rho}$

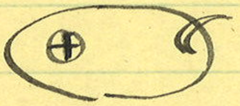
$\bar{\rho}_{☾} \sim 3.31 \quad , \quad \bar{\rho}_{\odot} \sim 1.41 \quad , \quad \frac{☾}{\odot} \sim \frac{1}{0.43}$

☾ effect a little more than twice that of ⊙.
The dominant tide is the <u>lunar</u> semi-diurnal.

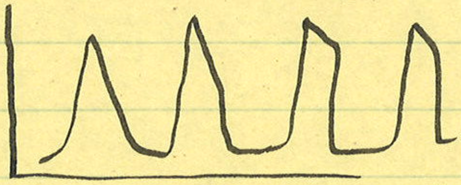
Can also ask: what is <u>relative</u> effect of ⊕ and ⊙ on <u>lunar</u> (solid body tides)?
Solar tides on ☾ ~ solar tides on ⊕

$$\frac{\oplus}{\odot} \sim \frac{M_{\oplus}/R_{\oplus}^3}{M_{\odot}/R_{\odot}^3} \sim \frac{200}{1}$$

Lunar tides dominated by ⊕, vy. large tides.
Nature of terrestial tides on ☾, since ☾ keeps same face toward ⊕ (a consequence of past <u>tidal friction</u>), dominant period <u>fortnightly</u>, eccentricity of orbit.



Recent interest: moonquakes are apparently being triggered by tidal strains



Moonquake energy release only 10^{11} erg/<u>yr</u>.
No evidence of tidal triggering on ⊕ (or at least very little evidence — not a major factor). Reasons: 1. diff. in interior properties, ☾ thicker lithosphere, ⊕ dominated by <u>tectonic</u> strains, 2. tides on ☾ 200· tides on ⊕.

The equilibrium tide (Newton). Consider spherical ⊕, no ☾, ⊙ in sky. Grav. equipot. surfaces are spheres. Consider equipot. surface with $r = a$.

Let $\phi_0(\underline{r})$ denote grav. pot. of ⊕

$$\phi_0(a\hat{r}) = -\frac{GM_{\oplus}}{a} \equiv \phi_0$$

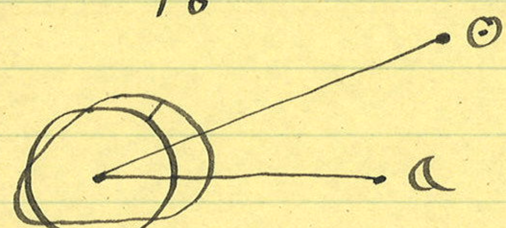
Grav. acceleration $\underline{g}(a\hat{r}) = -\nabla\phi_0(a\hat{r}) = -g\hat{r}$

Let $g \equiv |\underline{g}(a\hat{r})|$, value of gravity.

Now consider ⊕ + tide-generating bodies.

Assume ⊕ + oceans are rigid, no deformation

What is shape of equipot. surface with total potential ϕ_0?



Each pt. $\underline{r} = a\hat{r}$ moves to $\underline{r} + \underline{h}(\underline{r})$

Let $\phi(\underline{r}) \equiv$ total pot. (of undef. ⊕ + tidal pot.) at $\underline{r}$

On the displaced equipot. surface

$$\phi(\underline{r} + \underline{h}(\underline{r})) = \phi_0(\underline{r} + \underline{h}(\underline{r})) + V(\underline{r} + \underline{h}(\underline{r}))$$

$$= \phi_0(\underline{r}) + \underline{h}(\underline{r}) \cdot \nabla\phi_0(\underline{r}) + V(\underline{r}) + O(|\underline{h}(\underline{r})|^2)$$

Let $h(\underline{r}) \equiv \hat{r}\cdot\underline{h}(\underline{r})$. Also $\nabla\phi_0(\underline{r}) = +g\hat{r} = -\underline{g}$

Then

$$\phi(\underline{r}+\underline{h}(\underline{r})) = \phi_0(\underline{r}) - gh(\underline{r}) + V(\underline{r}) + \cdots$$

We want $\underline{h}(\underline{r})$ to be $\ni$ $\phi(\underline{r}+h(\underline{r})) = \phi_0$

Thus, on $r = a$,

$$h(a\hat{r}) = -\frac{1}{g}V(a\hat{r}),$$ tidal elevation of surface equipot. at pt. $a\hat{r}$

If solid $\oplus$ rigid, and ocean massless (no grav self attraction), and tid. pot. changed very slowly, ocean surface would always have shape of the slowly changing equipotential surface. Hence $h(a\hat{r},t)$ is called the equilibrium tide [ht.].

Now $g = \frac{GM_\oplus}{a^2}$. Thus

$$h(a\hat{r}) = \sum_{l=2}^{\infty} a\frac{M_☾}{M_\oplus}\left(\frac{a}{R}\right)^{l+1} P_l(\cos\alpha)$$

Let $\overline{R} \equiv$ mean $\oplus$ ☾ distance $\sim 3.844\cdot10^5$ km

Then

$$h(a\hat{r},t) = \sum_{l=2}^{\infty} K_l \left[\frac{\overline{R}}{R(t)}\right]^{l+1} P_l[\cos\alpha(t)]$$

changing dist. changing direction

$$K_\ell \equiv a \frac{M_☾}{M_\oplus}\left(\frac{a}{\bar{R}_☾}\right)^{\ell+1} \equiv a \frac{M_☾}{M_\oplus}\left(\bar{\xi}_☾\right)^{\ell+1}$$

here $\bar{\xi}_☾ \equiv$ <u>mean equatorial parallax</u>

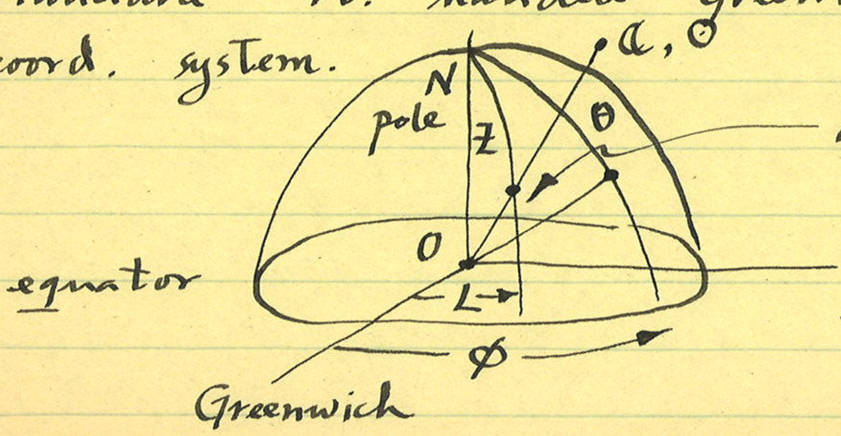
(This way dist. in solar system measured before development of radar). Term still in use.

$☾ \; K_2 = 35.785$ cm
$\odot \; K_2 = 16.427$ cm $\lesssim \frac{1}{2} ☾$

These are typical equil. tidal hts.
Now $K_\ell \sim \bar{\xi}^{\ell+1}$, $\bar{\xi}_☾ = 3422.6'' \sim \frac{1}{60}$ rad.
$\bar{\xi}_\odot = 8.79415'' \sim \frac{1}{23{,}000}$ rad.

$K_{3☾}$ tides $\sim \frac{1}{2}$ cm, difficult to observe.

Now introduce following change of variables.
Let θ, ϕ : colat, long of obs. pt.
$z(t), L(t)$: colat, long, of sub ☾ pt.
or sub $\odot$ pt.
Standard rt. handed Greenwich geocentric coord. system.



$\hat{r} : (\theta, \phi)$
$\hat{R} : (z, L)$

Spherical harmonic addition theorem

$$P_\ell(\hat{r}\cdot\hat{R}) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{r}) \overset{*}{Y}{}_\ell^m(\hat{R}) \quad \star$$

Here Y_ℓ^m is the fully normalized complex surface spherical harmonic

$$\int_\Omega Y_\ell^m(\hat{r}) \overset{*}{Y}{}_{\ell'}^{m'}(\hat{r})\, dA = \delta_{\ell\ell'}\, \delta_{mm'}.$$

Convenient to define <u>real</u> surface spherical harmonics, defining

$$Y_\ell^m = Y_\ell^{mc} + i\, Y_\ell^{ms}$$

Note $Y_\ell^m \sim e^{im\phi}$, $Y_\ell^m \sim \cos m\phi$, $Y_\ell^{ms} \sim \sin m\phi$

Making use of $Y_\ell^{-m} = (-1)^m \overset{*}{Y}{}_\ell^m$, rewrite $\star$

$$P_\ell(\hat{r}\cdot\hat{R}) = \frac{4\pi}{2\ell+1} \Big[Y_\ell^0(\hat{r})\, Y_\ell^0(\hat{R})$$

$$+ 2 \sum_{m=1}^{\ell} Y_\ell^{mc}(\hat{r})\, Y_\ell^{mc}(\hat{R}) + Y_\ell^{ms}(\hat{r})\, Y_\ell^{ms}(\hat{R}) \Big]$$

No negative m

Substitute : $$\overset{☾}{h}(\theta,\phi,t) = \sum_{\ell=2}^{\infty} \Big[a_\ell^0(t)\, Y_\ell^0(\theta,\phi) + 2\sum_{m=1}^{\ell} a_\ell^m(t)\, Y_\ell^{mc}(\theta,\phi) + b_\ell^m(t)\, Y_\ell^{ms}(\theta,\phi) \Big]$$

where
$$a_\ell^0 = \frac{4\pi K_\ell^{☾}}{2\ell+1}\left(\frac{\bar{R}}{R}\right)^{\ell+1} Y_\ell^0(z,L)$$

$$a_\ell^m = \frac{4\pi K_\ell^{☾}}{2\ell+1}\left(\frac{\bar{R}}{R}\right)^{\ell+1} Y_\ell^{mc}(z,L)$$

$$b_\ell^m = \frac{4\pi K_\ell^{☾}}{2\ell+1}\left(\frac{\bar{R}}{R}\right)^{\ell+1} Y_\ell^{ms}(z,L)$$

$R(t)$, $z(t)$, $L(t)$ are geocentric coords of ☾, ☉ : geocentric distance, colatitude, and longitude °E of Greenwich

If all but $\ell = 2$ are neglected

$$\overset{☾}{h}_2(\theta,\phi,t) = K_☾\left[\frac{\bar{R}}{R(t)}\right]^3 \Big[P_2(\cos\theta)\, P_2(\cos z(t)) + \frac{3}{4}\sin 2\theta \sin 2z(t) \cos(L(t)-\phi) + \frac{3}{4}\sin^2\theta \sin^2 z(t) \cos 2(L(t)-\phi) \Big]$$

better to write as on p. 22

Suppose for the moment that ☾, ☉ <u>fixed</u> in space: $R(t) =$ const, $z(t) =$ const, but $L(t) = 2\pi t / 24$ hrs, due to ⊕ rotation upon its axis.

Then $\quad a_\ell^m(t) \sim \cos mL = \cos \dfrac{2m\pi t}{24 \text{ hrs}}$

and

$$b_\ell^m(t) \sim \sin mL = \sin \frac{2m\pi t}{24 \text{ hrs}}$$

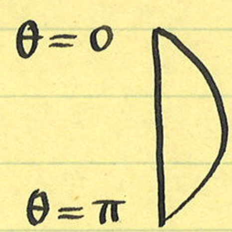
Actually 1 sidereal day = 23 h 56 m 4 s.

The order m of the spherical harmonic det. the freq. with which the equil. tide rises and falls. One defines the species of any tidal component to be

$$\text{species } t \equiv 24 \text{ hrs} / m$$

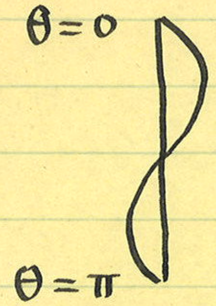
There are three principal ($\ell = 2$) tidal species.

1. semi-diurnal species, $m = 2$, $t = 12$ hrs
   two tidal maxima per lunar or solar day. Shape of equilibrium tide on Earth surface $\sim P_2^2 \sim \sin^2\theta$



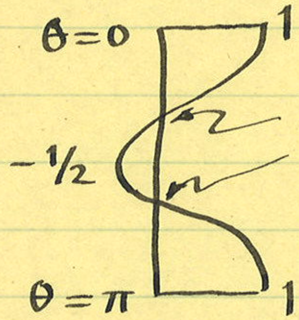
zero at poles, max. at equator

2. diurnal species, $m = 1$, $t = 24$ hrs.
   shape of equil. tide $\sim P_2^1 \sim \sin 2\theta$



zero at poles and at equator, max at lat 45° N, S.

3. long term species, $m=0$, $t=\infty$
actually since ☾, ☉ not stationary in sky, these tides do vary slowly
largest is lunar fortnightly ~ 2 wk. period.
shape of long term species ~ P_2^0



nodes at 35°16′ N, S lat

The $l=3$ terms (down by ~ 1/60 : $\frac{1}{2}$ cm high) lead to all three above species, but also to <u>terdiurnal</u> species, $m=3$, $t=8$ hr. Both terdiurnal ocean + Earth gravity and strain tides have been detected.

In general, $Y_l^m = 0$ at the poles unless $m=0$. Thus only long term species are non-zero at the poles. Slichter (UCLA) has for several years monitored the predominantly 2 wk. gravity tide at S pole.

Harmonic expansion of the tidal potential. Consider only $l=2$

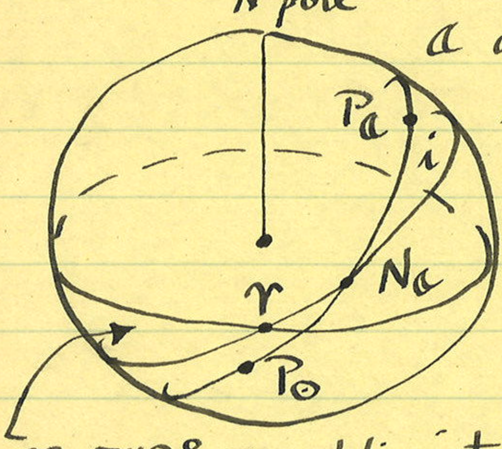
$$h_2^{☾}(\theta,\phi,t) = K_{☾}\left[\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right]\left[\left(\tfrac{R}{\bar{R}}\right)^3\left(\tfrac{3}{2}\cos^2 Z - \tfrac{1}{2}\right)\right]$$

$$+ K_{☾}\left[\sin 2\theta\right]\left[\tfrac{3}{4}\left(\tfrac{R}{\bar{R}}\right)^3 \sin 2Z \cos(L-\phi)\right]$$

$$+ K_{☾}\left[\sin^2\theta\right]\left[\tfrac{3}{4}\left(\tfrac{R}{\bar{R}}\right)^3 \sin^2 Z \cos 2(L-\phi)\right]$$

Each species is $K_{☾}$ [shape][astron. factor]

<u>Very</u> brief review of principal features of ☾, ☉ motions in a <u>geocentric</u> frame. Look on <u>celestial sphere</u>: sky as viewed from center of ⊕.

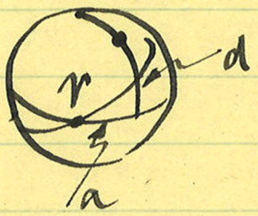


23.542° ≡ obliquity of ecliptic

♈ ≡ vernal equinox or Aries; $N_{☾}$ ≡ lunar node;

$i \equiv 5.145° \equiv$ inclination of ☾ orbit
$P_☉ , P_☾ \equiv$ solar and lunar perigee

Posns on celestial sphere commonly given as :



$d \equiv$ declination, measured from celestial equator
$a \equiv$ right ascension, measured from Aries (♈) along celestial equator

Motion of ☉ much simpler than that of moon. Basically an elliptical orbit in the ecliptic plane (obliquity $\equiv 23.542°$) So called period of solar declination is 365.242199 mean solar days (GMT is kept in mean solar days).

But orbital elements change very slowly due primarily to planetary (mostly Jupiter and Venus) perturbations. Most rapidly varying is advance of solar perigee at rate of $1.72°$/cent. Period = 20,940 yrs.

Motion of ☾ is very complex, because of its large mass rel. to ⊕, and its relative proximity. Complication traditionally divided into secular and periodic perts.

secular: slow changes of orbital elements. Such terms are really periodic; called secular, since they arise as linear & quadratic terms in the celestial mechanics analysis ($\sin \sigma t \sim \sigma t + \ldots$). Secular perts. suffice for descr. of ☉ motion.

periodic: ☾ has many relatively rapid periodic perts.; the inequalities are so large they are not best expressed as changes of orbital elements.

Period of lunar declination (defined by orbital speed of mean ☾) is 27.321582 msd.

Two most rapid secular perts. are:

1. advance of lunar perigee
   8.847 years
2. regression of lunar nodes
   18.613 years (clockwise, along ecliptic)

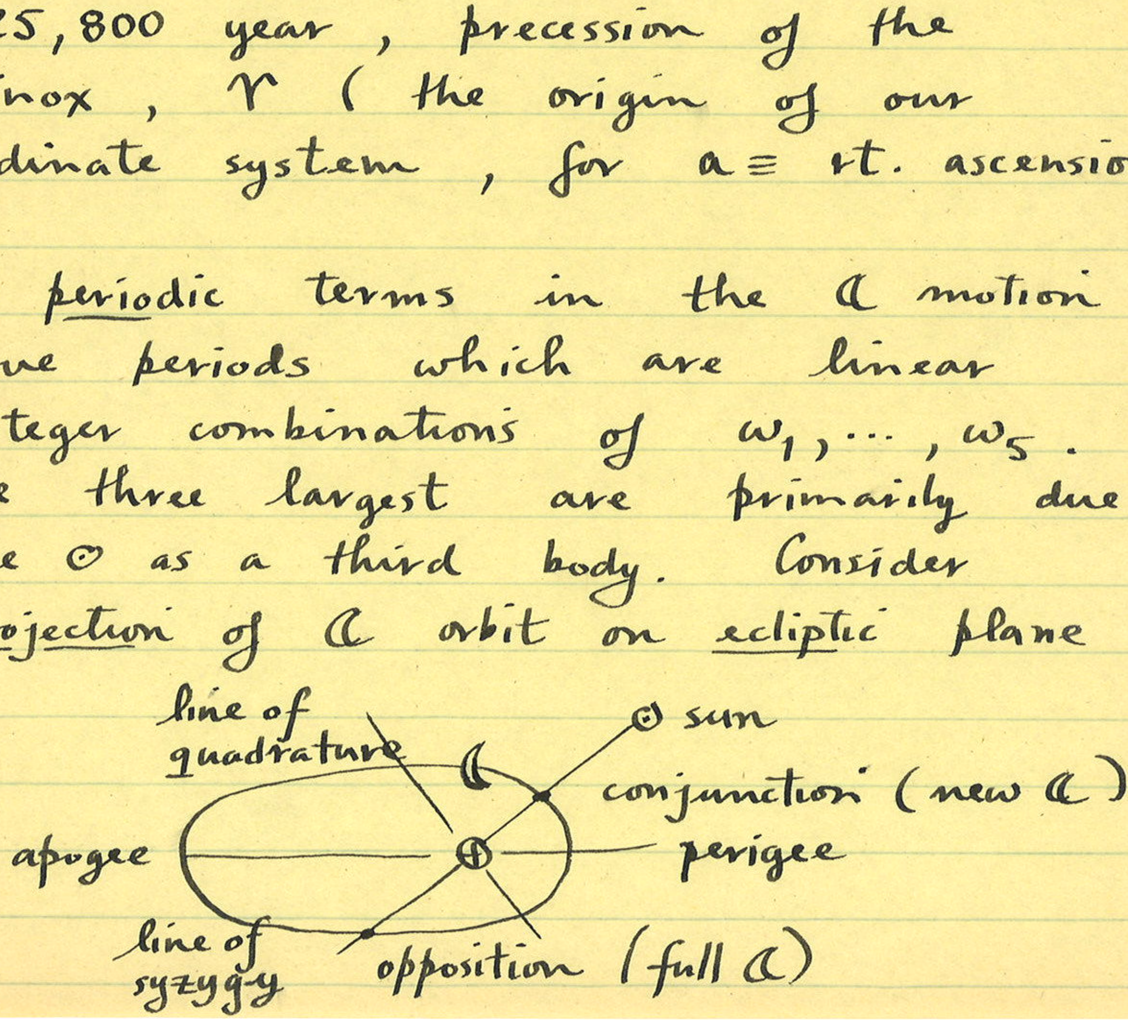
Both these are about 99.99 % due to presence of ☉ as a third body, residual due to other planets, oblateness of ⊕, ☾ and GTR (latter correction ≈ measurement error)

There are <u>five</u> fundamental periods in geocentric ☾, ⊙ motion. These also fundamental to tidal theory. Denote corr. angular freq by $\omega_1, \omega_2, \ldots, \omega_5$.

$2\pi/\omega_1 = 27.321582$ msd ☾ declination
$2\pi/\omega_2 = 365.242199$ msd ⊙ declination
$2\pi/\omega_3 = 8.847$ tropical yrs ☾ perigee
$2\pi/\omega_4 = -18.613$ " ☾ node (regresses)
$2\pi/\omega_5 = 20{,}940$ " ⊙ perigee

Note: next highest <u>fundamental</u> period is 25,800 year, precession of the equinox, ♈ (the origin of our coordinate system, for $a \equiv$ rt. ascension!)

The <u>periodic</u> terms in the ☾ motion have periods which are linear integer combinations of $\omega_1, \ldots, \omega_5$. The three largest are primarily due to the ⊙ as a third body. Consider <u>projection</u> of ☾ orbit on <u>ecliptic</u> plane



Effects best visualized as effect of $\odot$ on the effective $M_\oplus$

1. annual inequality : $M_\oplus$ minimum at solar perigee, month then the longest.
   Period of this term $2\pi / \omega_2 - \omega_5$ or 365.260 days (msd).
2. evection : $M_\oplus$ effective has a ~ monthly term due to ellipticity of lunar orbit, this the largest perturbation of all, ☾ can be ahead or behind mean ☾ by ~ 2 hrs or 1°. Has considerable effect on time of eclipses, max. anomaly at conjunction and opposition (time of eclipses), known to Ptolemy.
   Period of this term $2\pi / \omega_1 - 2\omega_2 + \omega_3$ or 31.812 msd. Why this period?

These are radial effects of $\odot$, $\exists$ also periodic terms due to tangential pull of $\odot$. Largest called

3. variation : a ~ fortnightly effect minimum at syzygy and quadrature, amplitude ~ ½° or 1 hr, zero at syzygy $\rightarrow$ little

effect on eclipses, not observed until careful obs. of T. Brahe.
Period of this term $2\pi / 2(\omega_1 - \omega_2)$ or 14.765 days.

Each tidal species is of the form

$$K_{\mathbb{C},\odot} [\text{ shape }][\text{ astronomy } (t)]$$

For

$$m=0: \quad [\,] = \left[(R/\bar{R})^3 \left(\tfrac{3}{2} \cos^2 z - \tfrac{1}{2} \right) \right]$$

$$m=1: \quad [\,] = \left[\tfrac{3}{4} (R/\bar{R})^3 \sin 2z \cos (L - \phi) \right]$$

$$m=2: \quad [\,] = \left[\tfrac{3}{4} (R/\bar{R})^3 \sin^2 z \cos 2 (L - \phi) \right]$$

By inspection, none is simple harmonic, or even periodic. If $\oplus$ and oceanic tidal response is idealized as <u>linear</u>, we know it is convenient to work with simple harmonic inputs. We could just use the F.T. of each of the [].

Alternatively we may employ the so-called harmonic expansion method, first suggested by Lord Kelvin.

One seeks, algebraically to write each of the factors [] in the form

$$[\text{astronomy}\,(t)] = \sum_j C_j \cos(\sigma_j t - m\phi + \theta_j) \quad \bigstar$$

where

$$\sigma_j = m\omega_{☾} + \sum_{i=1}^{5} a_i \omega_i \,, \quad a_i \ \underline{\text{integers}}$$

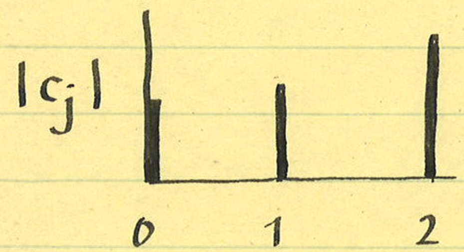
here $\quad \omega_{☾} \equiv \Omega - \omega_1$, $2\pi/\omega_{☾} = 1$ mean lunar day $= 24\,h\ 50m\ 28s$

Use of $\omega_{☾}$ <u>convenient</u> as ☾ tides largest. Can be qualitatively understood as a modulation of the three basic tidal species "carrier frequencies" 0, $\omega_{\odot,☾}$, $2\omega_{\odot,☾}$. One speaks of line splitting.
In a sense, non-linearity of eqns of celestial mechanics $\rightarrow$ sum and difference frequencies.

Each of the sums ★ is the result of celestial mechanics + prodigious algebra. Each equil. tidal species is split into a number of tidal lines. Ampl. spectrum of equilibrium tide is thus a <u>line</u> <u>spectrum</u>, but of the following rather peculiar nature. In principle, the number of terms in the sum is <u>infinite</u>. Line splitting is in principle an ∞ process.

In practice, can only ever achieve a finite sum approximation.
One speaks of the <u>resolution</u> of any particular harmonic development.

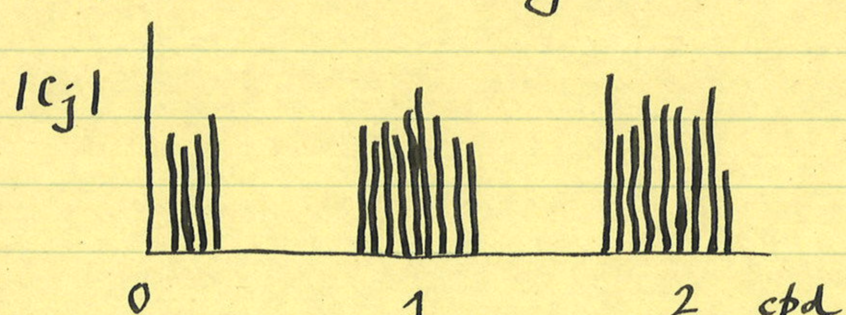
Results presented graphically as an equilibrium tidal line spectrum.
At 1 cpd resolution; a very crude approximation;



~ corresponds to ☾, ☉ still in sky

(mean solar day ≡ d)

At 1 cpm resolution, one sees <u>fortnightly splitting</u> of each of the above tidal species. Each line is now called a tidal <u>group</u>; minimum spacing between lines is ~ 2 wks;
$2\pi / 2\omega_1 = 13.661$ days.



At 1 cpy resolution, one sees <u>annual</u> splitting of the various tidal groups. Each line now called a tidal <u>constituent</u>.

Minimum spacing now ~ ½ yr ;
$2\pi / 2\omega_2 = 182.621$ days.

Harmonic expansion first suggested by Kelvin, carried out by G. Darwin ~ 1860's.
Carried out expansion to 39 terms and neglected any nodal splitting, i.e. worked at 1 cpy resolution. Smallest $c_j \sim 10^{-3}$. largest. Gave names to various constituents, e.g.

M_2 ☾ semi-diurnal
Sa ☉ annual
M_f ☾ fortnightly

A.T. Doodson 1921 Proc. Roy. Soc. carried out further to largest ~ 400 c_j, smallest $c_j \sim 10^{-4}$. largest, and found exact speed or frequency of each term, i.e. worked to solar perigee resolution, but truncated sum at ~ 400. Took him ~ 15 yrs of painstaking algebra. His notation (slightly modified) is convenient. Identify a tidal line by its Doodson numbers

$$(m, a_1, a_2, a_3, a_4, a_5)$$

(Actually Doodson added 5 ("positivizer") to a_i.

Table of few largest lines in equil tidal spectrum: Doodson, Darwin, speed, ☾ or ⊙, c_j

Long period

(0 2 0 0 0 0)	M_f	13.661 days	☾	.1564
(0 1 0 -1 0 0)	M_m	27.555 days	☾	.0825

Diurnal

(1 1 0 0 0 0)	K_1	1 sidereal day	☾, ⊙	.5305
(1 -1 0 0 0 0)	O_1	25.819 hrs	☾	.3769

Semi-diurnal

(2 0 0 0 0 0)	M_2	12.421 hrs	☾	.9081
(2 2 -2 0 0 0)	S_2	12 hrs	⊙	.4229

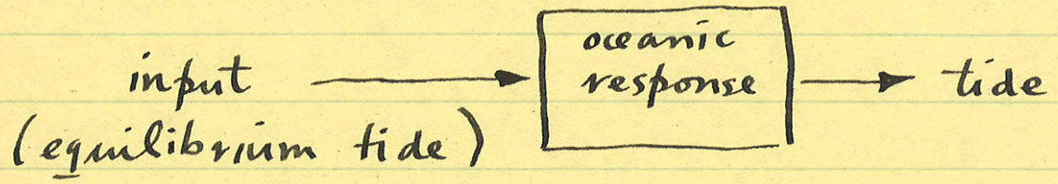
M_2 is largest. A peculiarity is K_1 at exactly 1 sidereal day, a lunisolar tide (has both ☾, ⊙ contribution).

Now have a complete descr. of cause of tides, or at least the <u>major</u> cause. We now turn to res<u>ponse</u> of <u>Earth</u> + <u>oceans</u>.

First a subject of commercial importance <u>Tidal</u> <u>prediction</u> (especially at ports)
We here model the oceans as a <u>black box</u>.
Munk and Cartwright 1966 distinguish three methods of tidal prediction.

<u>Nonharmonic methods</u>: Connection with ☾ so obvious that long before any theory $\exists$, satisfactory rule of thumb predictions were made. Methods considered gainful family possessions (father $\to$ son). Methods undivulged. Typically some form of <u>regression</u>.
Describe tides by a few tidal elements (time, ht. of high, low water, etc.) and relate to certain astron. variables (phase of ☾, declination of ☾, etc.). Tricky part: which variables to use.

<u>The harmonic method</u>: developed by Lord Kelvin and G. Darwin beginning ~ 1867. A method with a physical basis: <u>assume</u> that oceanic response at any port is <u>linear</u> and <u>time-invariant</u>. Then



Let us change notation slightly. Let equilibrium tide at $(\hat{r}, t)$ be denoted by $\bar{\xi}(\hat{r}, t)$ (previously $h(\hat{r}, t)$), and let actual tide be $\xi(\hat{r}, t)$, with no overbar.

This the <u>reason</u> for the harmonic expansion. Consider a single port $\hat{r}$, suppress argument. Can write, for any $\hat{r}$,

$$\bar{\xi}(t) = \sum_j \bar{C}_j \cos(\sigma_j t + \bar{\theta}_j)$$

where the $\sigma_j = m\omega_{\mathbb{C}} + \sum a_i \omega_i$, the <u>speeds</u> in the <u>equil.</u> tidal line spectrum.

Actually, in applying the method, one makes more than assumption of <u>linearity</u>, assume also <u>in practice</u>

1. $\xi(\hat{r}, t)$ depends only on $\bar{\xi}(\hat{r}, t)$, i.e. no other $\hat{r}$ (not a necessary restriction)
2. no <u>extreme</u> resonances amplify neglected σ_j

Then $\xi(t)$ must be of form $\bar{\xi}(t) \to \boxed{\text{port}} \to \xi(t)$

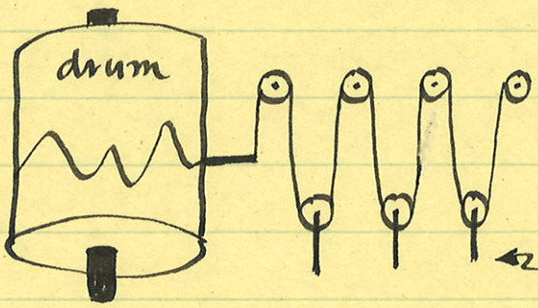
$$\xi(t) = \sum_j C_j \cos(\sigma_j t + \theta_j) \quad , \ \underline{\text{same } \sigma_j}$$

For any port, can determine coefficients C_j, θ_j from analysis of <u>past</u> tidal records, by method of least <u>squares</u>; once these coefficients have been determined, the tide can be predicted for any future time t.

Recall Darwin carried out harmonic expansion to 39 terms. By early 1900's method well established, today in common use. Reason Doodson felt it was worth 15 yrs to go to ~ 400 terms. Many residual tides showed strong components after removal of Darwinian components.

i.e. Doodson number

Kelvin invented mechanical machines for harmonic analysis of past tidal records (dial a speed, trace the curve, determine c_j, θ_j), also invented a similar machine for drawing predicted tidal curves. Samples of both in Smithsonian. Completely mechanical analog computers



$c_j \cos(\sigma_j t + \theta_j)$

Reset the 39 lower pulleys for any port "run off the tides for 1 yr in about 4 hrs".

Harmonic method, with the 1921 Doodson harmonic expansion is still used almost universally today for practical tidal prediction, mostly on modern computers.

In practice, one confronts the following problem: at many ports, only a few yrs of tidal data are available, and it is not practical to resolve terms closer than ~1 cpy. We are forced back to Darwin constituents (i.e. ignore $\omega_3, \omega_4, \omega_5$). Accepted procedure: let c_j, θ_j change from year to year, and recompute every year. Variation of c_j, θ_j is principally nodal, 18.61 yrs. The method is not then really harmonic at all.

The response method (Munk and Cartwright, 1966, note: pre FFT). Only purpose of harmonic expansion was to select, once and for all, the important frequencies to use in the harmonic analysis. One could have e.g. used all freqs. $\sigma_j = m\omega_a + \sum a_i \omega_i$, $-5 \le a_i \le 5$; this is $5 \cdot 10^{11}$ freqs., i.e. wheels in Kelvin's engine. Munk and Cartwright advocate essentially this, using computers. They say write

$$\xi(t) = w(t) * \bar{\xi}(t)$$

Compute $\bar{\xi}(t)$ at any $\hat{r}$; find response

function $w(t)$ of port by analyzing past records. Then $\xi(t)$ can be predicted.
Advantage: all astronomical frequencies automatically built in.

To what extent do observed tidal amplitudes + phases c_j, θ_j resemble the equil. factors $\bar{c}_j, \bar{\theta}_j$.
Coastal resonances can produce admittances of ~ 10.
But even mid-ocean, there is no resemblance; tides are a truly resonant phenomenon.
For example at Honolulu, semi-diurnal admittance $c_j / \bar{c}_j = \frac{1}{2}$ to $1\frac{1}{2}$, phase lag ~ 2 radians.

Note in the figures from Munk and Cartwright: the meteorological continuum

1. energy between tidal lines
2. some non-coherent energy at tidal freq.

Due to pressure fluctuations and solar heating and cooling. Note tidal cusps

Linear tidal prediction works well for most ports, for others response is significantly non-linear, generally associated with drag on shallow bottoms or thru narrow passages.

Example: Gulf of California

dominant tides have a <u>six</u> hour periodicity.

Tiburon Island

There are famous tidal currents in straits near Tiburon Island.

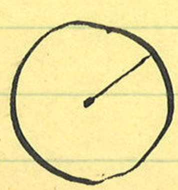
Now turn to a separate topic. <u>The solid Earth response</u>.

Solid $\oplus$ response is <u>equilibrium</u>, to a very good approximation.

${}_0S_2$ period ~ 54 minutes; semi-diurnal species $\sim$ 12 hrs $\gg$ 54 min.

We shall assume that:

1. $\oplus$ has <u>no oceans</u>
2. solid $\oplus$ is <u>spherically symmetric</u>
3. $\oplus$ is <u>non-rotating</u>



radius $r = a$.

For elastic-gravitational purposes, three parameters

$\rho_0(r)$ density
$\kappa(r)$ bulk modulus, incompressibility
$\mu(r)$ rigidity

Note: $\kappa(r)$ is isentropic, by assumption. Both are in situ.
These related to P, S elastic wave velocities by $\alpha(r)$, $\beta(r)$, $\alpha^2 = \dfrac{\kappa + 4/3\,\mu}{\rho_0}$, $\beta^2 = \mu/\rho_0$.

A realistic $\oplus$ model has a solid inner core $r \simeq 1230$, a fluid outer core $r \simeq 3484$ ($\mu(r) \equiv 0$), and a solid mantle + crust, $a = 6371$ km.

The tidal potential, at surface $r = a$, is of the form

$$\frac{1}{g} V(\hat{r}, t) \equiv \frac{1}{g} V(a\hat{r}, t) = \sum_{\ell=2}^{\infty} a_\ell^0(t)\, Y_\ell^0(\hat{r})$$
$$+ 2 \sum_{m=1}^{\ell} a_\ell^m(t)\, Y_\ell^{mc}(\hat{r}) + b_\ell^m(t)\, Y_\ell^{ms}(\hat{r})$$

For any $r \leq a$.

$$V(r\hat{r},t) = g \sum_{l=2}^{\infty} \left(\frac{r}{a}\right)^l \Big[a_l^0(t)\, Y_l^0(\hat{r})$$

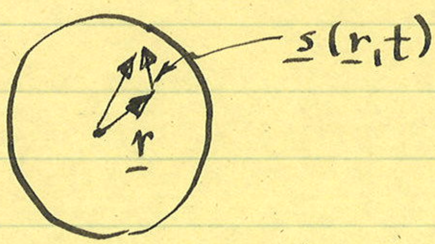
$$+ 2 \sum_{m=1}^{l} a_l^m(t)\, Y_l^{mc}(\hat{r}) + b_l^m(t)\, Y_l^{ms}(\hat{r}) \Big]$$

Write this in the form

$$V(r\hat{r},t) = \sum_{l=2}^{\infty} V_l(r\hat{r},t)$$

Each $V_l(r\hat{r},t)$ is a <u>solid spherical harmonic</u> of <u>degree l</u>.

The associated tidal force $T(r\hat{r},t) = -\nabla V_l(r\hat{r},t)$ acts at every point $\underline{r} = r\hat{r}$ within the Earth; this causes the Earth to deform. A particle originally at $\underline{r}$ will move to $\underline{r} + \underline{s}(\underline{r},t)$



Consider the <u>radial component</u> of displacement $u(r\hat{r},t) = \hat{r}\cdot\underline{s}(r\hat{r},t)$

Assume <u>only</u> that

1. ⊕ model is <u>spherically symmetric</u>
2. tidal response is <u>linear</u>
3. tidal response is <u>equilibrium</u>.

Then
$$u(\underline{r},t) = \sum_{l=2}^{\infty} u_l(\underline{r},t)$$

$$u_l(\underline{r},t) = -h_l(r)\,\frac{1}{g}\,V_l(\underline{r},t)$$

↑ fcn. only of radius r

This is physically obvious: spherically symmetric ⊕ → $Y_l^m(\hat{r})$ input → $Y_l^m(\hat{r})$ output. Equilibrium → no convolution in time domain. Not $h_l^m(r)$ by <u>Shur's lemma</u>, or since $\hat{z}$ axis arbitrary.

At surface $r=a$, $u_l(a\hat{r},t)$ = vertical elevation of solid Earth

$$u_l(a\hat{r},t) = -h_l(a)\,\frac{1}{g}\,V_l(a\hat{r},t)$$

$h_l(a) \equiv h_l$, first <u>Love number</u> of degree, introduced by Love (1911)

Since $V_3, V_4 \ll V_2$, $h_2 \equiv h$ is most important

$$u(a\hat{r},t) \approx h\left[-\frac{1}{g}V_2(a\hat{r},t)\right]$$

A constant (h) times the equilibrium tide height.
The Love number h_ℓ is a function of the Earth model $\rho(r)$, $\kappa(r)$, $\mu(r)$, $0 \le r \le a$. Easily determined numerically, find response to a degree ℓ solid spherical harmonic.
If $\oplus$ rigid, $h \equiv 0$, no deformation.
For realistic Earth models $h \approx .609$
Note $[h] = 0$, $h \approx O(1)$.

The deformation will alter the gravitational self-potential of the $\oplus$

Before deformation: in $0 \le r \le a$

$$\phi_0(\underline{r}) = -\frac{G}{r}\int_0^r \rho(r)\, r^2\, dr$$

After deformation, total potential at a point, $0 \le r \le a$, $\underline{r} = r\hat{r}$

$$\phi_0(\underline{r}) + V(\underline{r}, t) + \phi_1(\underline{r}, t)$$

due to defm. of $\oplus$

Now spherical symmetry, linearity, equilibrium response imply

$$\phi_1(\underline{r},t) = \sum_{l=2}^{\infty} \phi_{1l}(\underline{r},t)$$

$$\phi_{1l}(\underline{r},t) = k_l(r)\, V_l(\underline{r},t)$$

$\uparrow$ fcn. only of radius r

In particular at $r = a$, $\phi_{1l}(a\hat{r},t) = k_l(a) V_l(a\hat{r},t)$
$k_l \equiv k_l(a)$, <u>second</u> Love number of degree l.

Once again $k_2 \equiv k$ is most important

Note: outside the $\oplus$,

$$\phi_1(\underline{r}) = \sum_{l=2}^{\infty} \left(\frac{a}{r}\right)^{l+1} k_l\, V_l(a\hat{r},t)$$

Falls off like $(a/r)^{l+1}$, must match at surface, $\nabla^2 \phi_1(\underline{r},t) = 0$ <u>outside</u>.

Total potential inside, approximately (neglect $l>2$)

$$\phi_0(\underline{r}) + (1+k)\, V_2(\underline{r},t).$$

$k(r)$ here.

For a rigid $\oplus$, $k = 0$

For a realistic $\oplus$, $k \simeq .300$

Third Love number introduced by Shida 1912.

Consider the <u>tangential displacement</u>

$$\underline{s}(\underline{r},t) = \hat{r}\hat{r}\cdot\underline{s}(\underline{r},t) + \underline{v}(\underline{r},t)$$
$$= \hat{r}\,u(\underline{r},t) + \underline{v}(\underline{r},t), \text{ say}$$

can be shown that, subject to the same assumptions,

$$\underline{v}(\underline{r},t) = \sum_{l=2}^{\infty} \underline{v}_l(\underline{r},t)$$

$$\underline{v}_l(\underline{r},t) = -\,l_l(r)\;\nabla_1\left[\tfrac{1}{g}V_l(\underline{r},t)\right]$$

here $\nabla_1 \equiv$ surface gradient on $\Omega_1 \equiv$ unit sphere

$$= \hat{\theta}\,\partial_\theta + \hat{\phi}\,\frac{1}{\sin\theta}\,\partial_\phi$$

$$\nabla = \hat{r}\,\partial_r + \frac{1}{r}\nabla_1$$

On $r = a$, $l_l(a) \equiv l_l$, third Love number of degree l

As before, l_2 often denoted simply l.

For realistic Earth models, $l \simeq .085$.

Love numbers h_l, k_l, l_l, $l \geq 2$: their importance is that any <u>measurement</u> of tidal response made on ⊕ surface depends <u>only</u> on these numbers.

The Love number $h_l(r)$, k_l, $l_l(r)$, $l \geq 2$, considered as a fcn of radius, completely describe the <u>elastic-gravitational response</u> to a <u>static</u> imposed external tidal potential. It is easy to see the result of removing the two most restrictive assumptions, sphericity + <u>quasi-</u>

static or equilibrium.

If model spherically symmetric, but response not equilibrium, then Love numbers are frequency dependent $h_\ell(\sigma)$, $k_\ell(\sigma)$, $h_\ell(\sigma)$.
Current interest in looking for such a frequency dependence near $\sigma = \Omega$, due to dynamical effects of the <u>fluid core</u>; this is essentially a Rossby wave resonance which we have no time to discuss.

If response is $\sim$ equilibrium, but <u>not</u> spherically symmetric, then

tidal potential $Y_\ell^m(\hat{r}) \rightarrow$ response <u>all</u> $Y_{\ell'}^{m'}(\hat{r})$

If defined in the obvious way, Love numbers would be dependent on $\hat{r}$
$h_\ell(\hat{r}), k_\ell(\hat{r}), l_\ell(\hat{r})$.

If <u>both</u> assumptions removed, $h_\ell(\hat{r},\omega)$, $k_\ell(\hat{r},\omega)$, $l_\ell(\hat{r},\omega)$.

<u>Love numbers</u> a convenient way of summarizing some of elastic-grav. properties of Earth.

Computation of Love numbers: linearized equations and b.c. Points of interest here for fluid dynamicists:

1. use of displacement $\underline{s}(\underline{r},t)$, not velocity $\partial_t \underline{s}(\underline{r},t)$
2. inclusion of self-gravitation
3. influence of initial stress $\underline{\underline{T}}_0(\underline{r}) = -p_0(\underline{r})\,\underline{\underline{I}}$, where $\nabla p_0 + \rho_0 \nabla \phi_0 = 0$; conv. elastic stress-strain relation is <u>at a particle</u>; equation comes from linearization of <u>Eulerian</u> law.

$$\rho_0 \partial_t^2 \underline{s} = -\rho_0 \nabla \phi_1 - \rho_1 \nabla \phi_0 - \nabla(\underline{s} \cdot \rho_0 \nabla \phi_0) + \nabla \cdot \underline{\underline{T}} - \rho_0 \nabla V$$

applied tidal pot.

$$\rho_1 = -\nabla \cdot (\rho_0 \underline{s}) \qquad \text{continuity}$$

$$\nabla^2 \phi_1 = 4\pi G \rho_1 \qquad \text{Poisson}$$

$$\underline{\underline{T}} = \kappa(\nabla \cdot \underline{s})\,\underline{\underline{I}} + 2\mu\left[\underline{\underline{\nabla s}} + (\underline{\underline{\nabla s}})^T\right]$$

boundary conditions

$\hat{r} \cdot \underline{\underline{T}}$ cont. ; $= 0$ on $r = a$, free surface

ϕ_1, $\hat{r} \cdot \nabla \phi_1 + 4\pi G \rho_0 \hat{r} \cdot \underline{s}$ cont.

Quasi-static $\rightarrow \partial_t \equiv 0$.
Now substitute

$$\underline{s} = \underbrace{\hat{r}\,u + \nabla_1 v}_{\text{spheroidal}} - \underbrace{\hat{r} \times \nabla_1 w}_{\text{toroidal}}$$

Find $w(\underline{r}) = 0$, and a sixth order ordinary linear diff. eq. for $u_\ell^m(r)$, $v_\ell^m(r)$, where

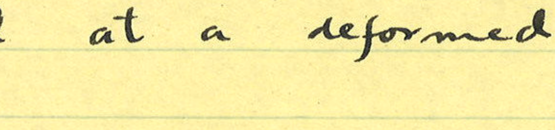
$$\underline{u}(r\hat{r}) = \sum u_\ell^m(r)\, Y_\ell^m(\hat{r})$$
$$v(r\hat{r}) = \sum v_\ell^m(r)\, Y_\ell^m(\hat{r})$$

This solved, subject to b.c., for $h_\ell(r)$, $k_\ell(r)$, $\ell_\ell(r)$, $\ell \geq 2$.

<u>Measurement</u> of <u>Love numbers</u>, by <u>surface</u> measurements, in principle.

Consider value of grav. pot. observed by an observer situated on <u>deformed</u> ⊕ surface

☾ or ☉



Point $\underline{r}$ moves to $\underline{r} + \underline{s}(\underline{r}, t)$. <u>Total</u> potential at a deformed position is

$$\phi(\underline{r} + \underline{s}) = \phi_0(\underline{r} + \underline{s}) + V(\underline{r} + \underline{s}) + \phi_1(\underline{r} + \underline{s})$$

$$= \phi_0(\underline{r}) + \underline{s} \cdot \nabla\phi_0(\underline{r}) + V(\underline{r}) + \phi_1(\underline{r}) + O(|\underline{s}|^2)$$

$-g u(a\hat{r})$, since $\nabla\phi_0(a\hat{r}) = -g\hat{r}$

For simplicity, let us neglect all but $l = 2$; generalization is obvious

$$V(\underline{r}) \sim V_2(\underline{r})$$
$$u(\underline{r}) \sim u_2(\underline{r}) \;,\quad u_2(a\hat{r}) = -h\left[\tfrac{1}{g}V_2(a\hat{r})\right]$$
$$\phi_1(\underline{r}) \sim \phi_{12}(\underline{r}) \;,\quad \phi_{12}(a\hat{r}) = k\left[V_2(a\hat{r})\right]$$

Thus

$$\phi(a\hat{r} + s) = \phi_0(a\hat{r}) - hV(a\hat{r}) + V(a\hat{r}) + kV(a\hat{r})$$

Net change in grav. pot. at deformed surface is thus

$$\delta\phi(a\hat{r} + \underline{s}(a\hat{r},t), t) = [1 + k - h]\, V(a\hat{r}, t)$$

A gravitational potentiometer (hypothetical, non-realizable instrument) would enable measurement of the combination of Love numbers $[1 + k - h]$ (in general, of $[1 + k_l - h_l]$).

Lord Kelvin pointed out that the existence of the oceans might provide us with such an instrument. If:

1. ocean tides were equilibrium
2. oceans covered $\oplus$ surface uniformly (no need for cons. of H_2O correction
3. oceans of negligible ρ (oceanic loading

and self-gravitation neglected),
then: changing ocean surface will always conform to <u>geoid</u> ≡ equipotential surface. The vertical displacement of the geoid <u>relative to the ocean bottom</u> is exactly, neglecting $V_\ell(a\hat{r},t)$, $\ell > 2$.

$$\xi(a\hat{r},t) = [\underbrace{1+k}_{\substack{\text{displ.}\\ \text{relative to}\\ \text{c.o.m. of } \oplus}} - h]\left[-\frac{1}{g}V(a\hat{r},t)\right]$$

but ocean floor moves up

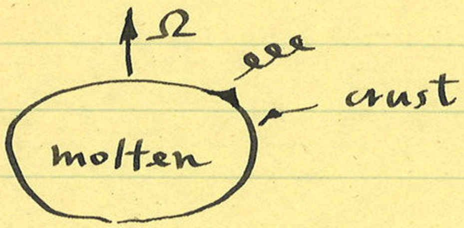
This should be evident; derivation exactly as before; note have neglected oceanic self-attraction and oceanic loading
Note: on a non-yielding $\oplus$, $h = k = 0$, same as before.

This argument led to first (pre-seismology) estimate of overall mantle rigidity by Lord Kelvin and George Darwin ~ 1880's. Laplace had previously suggested that long <u>period</u> tidal species (largest M_f ≡ lunar fortnightly) would be ≈ equilibrium, because 2 wks ≫ oceanic free periods ≈ 2·width of ocean basins / $\sqrt{gh}$.
On a yielding Earth, ht. of equil. tide $= [1+k-h]\cdot$ ht. on a rigid $\oplus$.
The Love numbers h, k depend on $\oplus$

model $\rho_0(r)$, $\kappa(r)$, $\mu(r)$. Idea of Kelvin: measure diminution factor $1+k-h$, find $\oplus$ rigidity.

Background: prevailing geological view of $\oplus$ interior at that time: <u>molten</u> interior, <u>thin</u> congealed outer crust. Reasons

1. $T(r) \nearrow$ as go down in mines: ask any Welch coal miner
2. volcanoes $\exists$
3. most important, $\varepsilon \approx \varepsilon_{hydrostatic}$



Now known that 3. is not <u>quite</u>, but almost, true: reason: long term rheology is far from elastic, high T creep. Such an Earth: $1+k-h \cong 0$, behaves as if all fluid.

Kelvin made following calculation (<u>we</u> use Love numbers to discuss his result).

1. linear elasticity theory
2. homogeneous incompressible sphere

$$\kappa(r) = \infty$$

$$\rho_0(r) = \rho_0 \quad , \quad \mu(r) = \mu$$

Define $\tilde{\mu} \equiv \dfrac{19}{2}\dfrac{\mu}{\rho g a}$, dimensionless rigidity

Then $\quad h = \dfrac{5/2}{1+\tilde{\mu}} \quad , \quad k = \dfrac{3/2}{1+\tilde{\mu}}$

Then

$$1 + k - h = 1 - \frac{1}{1+\tilde{\mu}} = 1 - \frac{1}{1 + \frac{19}{2}\frac{\mu}{\rho g a}}$$

For $\tilde{\mu} \sim \infty$, rigid $\oplus$, $h, k = 0$, $1+k-h = 1$

For $\mu = 0$, fluid $\oplus$, $h = 5/2$, $k = 3/2$, $1+k-h = 0$; <u>no</u> ocean bottom, ht. of tide above ocean bottom $\equiv 0$

In general $\quad 0 \leq 1+k-h \leq 1$, ht. on a yielding $\oplus$ $\leq$ ht. on a rigid $\oplus$. On a molten $\oplus$, ht. of M_f tide $\simeq$ zero.

G. H. Darwin undertook the analysis of then available tidal records; 14 British + Indian ports, total 66 yrs. of observation. Concluded, for M_f tide, $\quad$ Note: not an easy obs.

$$1 + k - h = 0.7 \pm 0.1$$

A reduction by $\sim 30\%$, $\quad$ M_f tides $\sim$ 1-2 cm.

$$\frac{1}{1 + \frac{19}{2}\frac{\mu}{\rho g a}} = 0.3 \quad , \quad \rho = 5.5 \text{ gm/cm}^3$$

$-\frac{1}{2}B_1^1$ <u>shape</u> $\propto P_2^0 \sim$ zero at mid-lat. $\bar{C}_j$ is .1564

Find $\mu \sim 8 \cdot 10^{11}$ dyne cm^{-2} $\sim \mu_{steel}$

Kelvin's famous remark: $\oplus$ is as rigid as steel.

Now of only historical interest: seismology gives much more detailed information on radial variation of elastic properties $\rho_0(r)$, $\kappa(r)$, $\mu(r)$. P and S wave $t-\Delta$ data + normal mode eigenfrequencies.

To turn problem around: what do we expect to see, on the equilibrium hypothesis?
$1+k-h = 1 + .300 - .609 = .691$.
Expect $\theta_j - \bar{\theta}_j = 0$, admittance $C_j / \bar{C}_j = .691$
No phase lag w.r.t. $\bar{\xi}(\hat{r}, t)$.

Most recent analysis of long period tides: Wunsch, Rev. Geophys. 5, 447-475 (1967). Finds deviations from equilibrium theory of up to ~ 50 % in amplitude and ± 30° in phase. Scale length of fluctuations is ~ 3000 km. Attributed to Rossby wave resonances; ocean basins have Rossby wave modes with periods $\gg$ 2·width of basins / $\sqrt{gh}$. We shall see this later.

How else can we measure Love numbers? Best modern measurements come from tidal gravimeters; measure small (1 part in ~10^7) changes in grav. acceleration on deformed Earth surface.
Total grav. attraction on deformed surface is $\nabla\phi(\underline{r} + s(\underline{r}, t))$, $\underline{r} \equiv a\hat{r}$.

We are interested in the change $\delta g(a\hat{r},t)$
$|\nabla\phi(a\hat{r}+\underline{s}(a\hat{r},t))| - |\nabla\phi_0(a\hat{r})|$ of the magnitude of the grav. accel.
Derivation is straightforward. Let
$\delta g(a\hat{r},t) \equiv$ change in (outward) grav. accel. Correct to first order,

$$\delta g(a\hat{r},t) = \sum_{\ell=2}^{\infty} \delta g_\ell(a\hat{r},t)$$

$$\delta g_\ell = -\left[1+\frac{2}{\ell}h_\ell - \frac{\ell+1}{\ell}k_\ell\right]\frac{\ell}{a}V_\ell$$

For $\ell = 2$:

$$\delta g(a\hat{r},t) \cong \delta g_2(a\hat{r},t)$$
$$\cong -\left[1+h-\frac{3}{2}k\right]\frac{2}{a}V(a\hat{r},t)$$

direct attr of ☾, ⊙: note − sign

lifting of gravimeter further from c.o.m. ⊕: note − sign

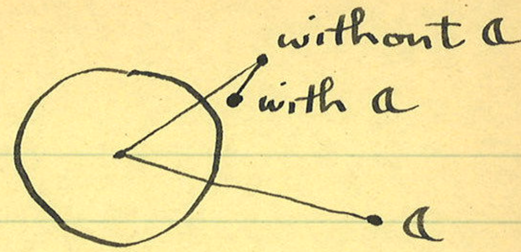
attr. of ⊕ tidal bulge: note + sign

when ☾, ⊙ overhead, $V(a\hat{r},t)$ negative

☾, ⊙

theory: 1.159

$1+h-\frac{3}{2}k$ called the gravimetric factor; may be inferred from measurements.
Question: can h, k be measured separately. Yes, in principle. Use a tidal tiltmeter to measure deflection of local vertical. Say ⊕ rigid. Then tidal attraction will deflect local vertical.



Recognized by Abel 1824

Let $\delta(a\hat{r},t) \equiv$ angular deflection of vertical

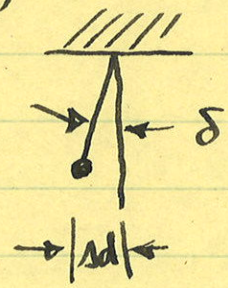
without proof: on a yielding ⊕, for $l=2$ only

$$\delta(a\hat{r},t) = [1+k-h]\,\delta_{\text{rigid}}(a\hat{r},t)$$

Note same comb. $1+k-h$ as for equil. ocean tides; this as expected, since local vertical is $\perp$ everywhere to geoid, i.e. to equil. ocean surface.

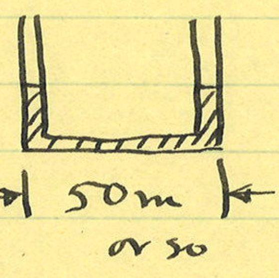
Tidal tiltmeters of two types:

1. pendulum



$\delta_{\text{rigid}} \sim 10^{-7}$ rad.
say length ~ 1 m.
$\Delta d \sim 10^{-7} d \sim 10^{-4}$ mm.

2. liquid level
a large spirit level; typically
utilizes a capacitive transducer



Both measurements very difficult, easily overwhelmed by thermal effects. Really good tiltmeter measurements $\equiv$ null set.

Third Love number l can be measured using tidal strain meters. Response of a single instrument depends location, orientation and both h and l. In principle, strain meters, by themselves, can determine both h and l. Example: at any θ, ϕ, comb. of NS and EW determines l uniquely.

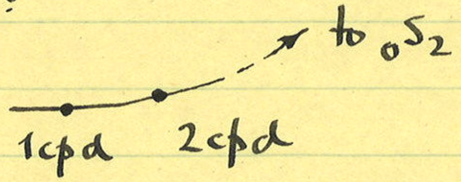
Earth tide research: the overall program.

Why measure $\oplus$ tides?

1. they are gross $\oplus$ data: input into any inversion; this was original motivation; no longer so important; so many other g.e.d. $\exists$.
2. examine validity + breakdown of assumptions. If we define $\underline{r}, \omega$ dependent "Love numbers" from our observations, then

Dependence on $\underline{r}$, regional Love numbers, related to regional differences in crust, upper mantle structure

Dependence on ω, deviations from equilibrium response of solid $\oplus$. For example:



expect response to ↗ for higer freq. on flat tail of resonance curve

Theory predicts resonance (very narrow)

near 1 cpd, due to dynamics of fluid core.
Would be very nice to measure phase lag due to tidal friction; determine % of tidal dissipation in solid ⊕ → constraint on solid ⊕ anelasticity at tidal periods.

Above program has been largely unsuccessful. Last 10 years: we now realize why. Contamination of measurements by oceanic tidal loading. Quantitatively can account for ~ 10 % of gravity tides and ~ 20 % of strain. Modern gravity meters are calibrated to $\lesssim 1\%$, laser strain meters are inherently well calibrated.

In absence of oceans, what do we expect for a gravimeter: $1 - h + \frac{3}{2}k = 1.16 \pm \lesssim 1\%$
the gravimetric factor
Theoretical phase lag, due to solid ⊕ anelasticity $\varepsilon = \tan^{-1}\frac{1}{Q}$. Assuming $Q_{\text{tides}} \simeq Q_{{}_0S_2}$
$\simeq 350 \pm 70$, find

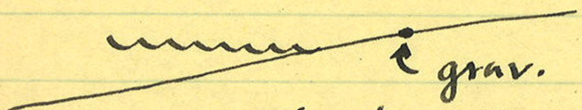
10^{-7} strain
0.1 cph

10^{-8} strain
1 cph

$\epsilon \sim 0.1°$ to $0.2°$, very small ⇒ > 99% of tidal dissipation must occur in oceans. This is an important assumption to check.

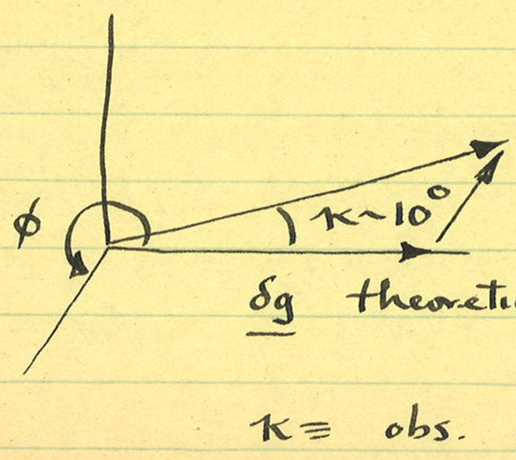
Many gravity tide obs. during IGY 1957 Measured $1 - h + \frac{3}{2}k$ ranging from 1.15 – 1.25 (too large for regional effects), phase lags as great as ± 5 – 10 %.

These discrepancies a result of <u>ocean loading</u>. <u>Three</u> effects on a gravimeter



1. direct attraction of H_2O
2. depression of ocean floor lowers gravimeter
3. load deforms solid ⊕

First two are largest, but third is <u>quantitatively</u> significant.
Convenient to divide loading effects into <u>local</u> and <u>global</u>. For coastal or mid-ocean (island) stations, the local effect can predominate. Example M_2 tide at Bermuda (an IGY station).



$\underline{\delta g}$ observed

$\underline{L} \equiv$ load tide $\equiv \underline{\delta g} - \underline{\delta g}_{theory}$

$\underline{\delta g}$ theoretical 1.16, phase ≈ 0 (lag w.r.t. equilibrium)

$\kappa \equiv$ obs. phase lag

$\phi \equiv$ obs. phase of Bermuda harbor tide
Observed that Bermuda load tide 180° out of phase with ocean tide in Bermuda harbor.
ϕ can be very different from zero because of ocean basin resonance.
Sign makes sense: high water $\rightarrow$ both direct H_2O attr. and downward depression give a downward acceleration and $\delta g(a\hat{r}, t)$ is change in outward accel.
For many coastal + island stations, one finds $\phi_{\text{load tide}} \approx \phi_{\text{local tide}} + 180°$.
For mid-continent stations, the global ocean tide is the whole story.

If global ocean tides were known, correction of $\oplus$ tides for the loading effect would be straightforward; can be reduced to numerical convolution with a Green's function. We will not have time here to discuss this.

The above mentioned $\oplus$ tide program (why measure $\oplus$ tides?) must be attacked in conjunction with problem of global oceanic tides. Conversely, can now add to list for why measure $\oplus$

tides: can be used to place constraints on global ocean tides.
This is probably, at the present time, the most important reason.

This brings us bluntly to the subject of:

Oceanic tides: from a global point of view.
An inherently dynamical phenomenon.
Traditional representation of the spatial configuration by means of cotidal lines.
Suppose that $\forall \underline{r}$ on surface $r=a$, we make a harmonic analysis of the tide
Consider a frequency σ, the corresponding periodic tide of freq. σ is of the form

$$\xi(\hat{r},t) = C(\hat{r}) \cos[\sigma t - \theta(\hat{r})]$$

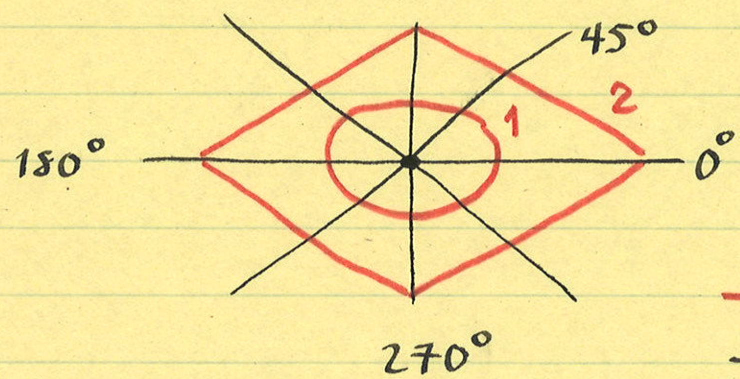
Cotidal lines are lines of constant phase $\theta(\hat{r}) =$ constant, typically θ° in degrees $0^\circ \le \theta^\circ \le 360^\circ$. Can also plot corange lines (lines of const. amplitude $C(\hat{r}) =$ const).

The characteristic feature of cotidal diagrams, whether constr. empirically or by numerical analysis is the breaking up into so-called amphidromic cells within which tides are turning about a central stationary point (the amphidromic point, or amphidrome)

Consider a simple example on a flat surface (Cart. coords. x, y) : superposition of two standing waves in quadrature with $\perp$ nodal lines.

$$\xi(x,y,t) = \sin kx \cos\sigma t + \sin ly \sin\sigma t$$

Cotidal lines are $\arg[\sin kx + i \sin ly] =$ const. Amphidrome, from which they radiate is $x, y = 0, 0$.



This pattern repeats in space

corange lines
cotidal lines

At amphidrome tide (of freq σ) is zero. Tide turns ccw in one cycle about that point.

Off shore tides are best known off Calif coast, mostly due to free capsule obs. of Munk, Snodgrass group at La Jolla.

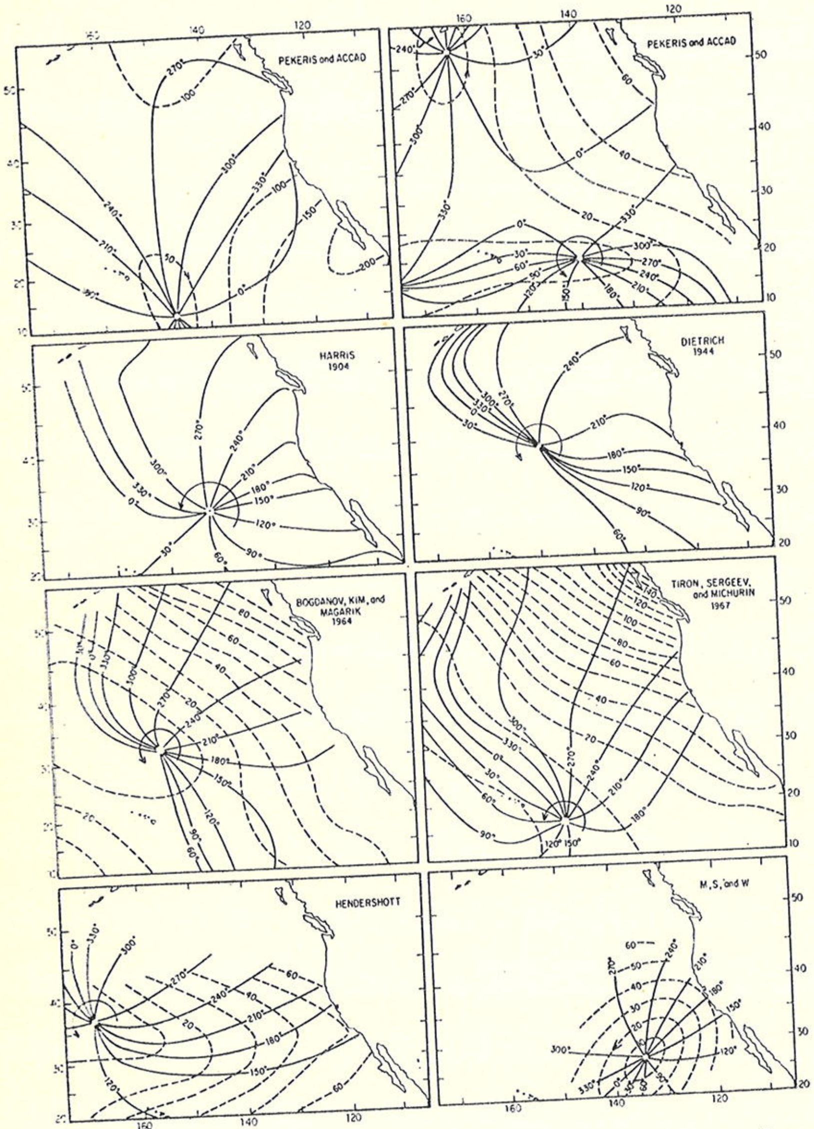


Figure 19. Comparative cotidal charts for M_2 elevations. Lines of equal Greenwich epoch, ${}^{\eta}G$, and of equal amplitude, ${}^{\eta}H$ cm, are shown by the solid and dashed curves. The left Pekeris-Accad chart is based on a relatively fine coastal grid (their map 1°A); the right chart for a coarser coastal grid (map 2°A).

zero phase is w.r.t.
the equilibrium tide

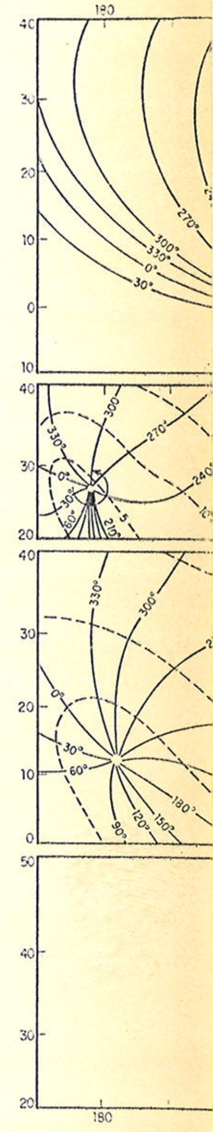


Figure 20. Compar

Fig. 3. Atlantic results of global numerical solutions of Laplace's tidal equations for the M2 tide by (left to right, top to bottom) Bogdanov and Magarik [1967], Hendershott [1972], Zahel [1970], Pekeris and Accad [1969],–frictionless, 2° mesh, Dietrich [1944],–an empirical map rather than a numerical solution, redrawn by Villain [1962], Pekeris and Accad [1969],–dissipative, 1° mesh.

feature discovered by the numerical tidalists.

Islands in the Indian Ocean are sufficiently frequent and evenly distributed that a rather good empirical map may be constructed (Figure 4). The salient feature of such a map is a very large region in the south central Indian Ocean within which the sea surface moves nearly synchronously. Every numerical map produces such an 'antiamphidrome.' The computations of *Pekeris and Accad* [1969] and *Hendershott* [1972] suggest that this region is nearly resonant at semidiurnal periods. Hendershott (unpublished) finds that by varying the mean depth by several hundred meters, computed $M2$ amplitudes may be changed by a multiplicative factor of 2 or 3. A very realistic prediction of Indian Ocean island semidiurnal tides may be made by proper empirical choice of the mean depth.

In the Pacific, the various numerical and normal mode models show the greatest divergence from one another and from Pacific island observations (Figure 5). This is perhaps not surprising in view of the size of this basin and the consequent diminution of the direct influence of coastal boundary conditions on mid-basin tides.

The empirical cotidal maps show a

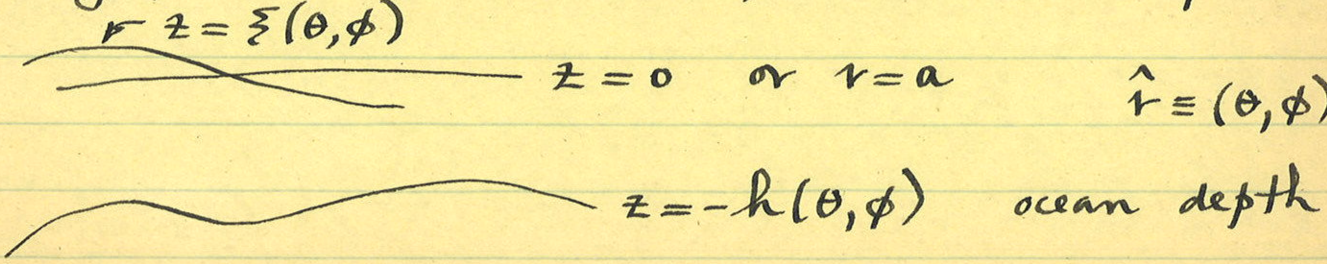
Established $\exists$ of ccw M_2 amphidrome off S. Calif. coast. Empirical cotidal charts seldom show corange lines: these constructed by straightforward extrapolation of coastal + scattered island observations. Most often quoted: those of Dietrich (1944) for M_2, S_2, K_1, O_1.
The reconstr. of Munk, Snodgrass, Wimbush is based on theoretical flat $\oplus$ solutions. Recently, finite difference methods are being employed.
Show examples Fig. 19 of MSW (1970) and Hendershott (1973) Fig. 3 to indicate divergence of opinions.

Since time of Laplace, theoretical work has been dominated by attempts to solve Laplace's Tidal Equations, hereafter LTE. We now very briefly indicate the assumptions underlying these equations; we shall not do so very systematically. We shall neglect or assume

1. solid $\oplus$ deformation (shall assume underlying solid $\oplus$ is rigid)
   study tides on a rigid globe.
2. assume oceans are incompressible, (this eliminates acoustic phenomena)
3. inviscid (we shall later add a bottom or skin friction)

4. homogeneous (this eliminates baroclinic phenomena); we shall restrict study to the surface or barotropic tide.
5. we shall neglect non-linear terms in Navier-Stokes eqn; consider small disturbances relative to a state of uniform rotation.
6. we shall work in the rotating frame but (this is strictly inconsistent) assume ⊕ spherical and assume gravity is constant $g(\hat{r}) \equiv -g\hat{r}$. Can be shown this gives rise to an error uniformly of $O(\varepsilon) \sim 1/300$.
7. we shall make the shallow water approximation, i.e. we shall assume that horizontal velocities are independent of r. The vertical velocity is a linear function of r or $z = r - a$.

At any point r, θ, ϕ in the oceans or alternatively z, θ, ϕ, where $z \equiv r - a$. Locally, we have the picture (looks flat)



Let $\xi(\hat{r},t)$ be the tide height

$h(\hat{r})$ ocean depth

These are fcns only of θ,ϕ or $\hat{r}$.

Tidal velocity field a function of $z,\hat{r}$

$$\underline{u}_{total}(z,\hat{r}) = \hat{r}\,w(z,\hat{r}) + \underline{u}(z,\hat{r})$$

i.e. let $\underline{u}(z,\hat{r})$ be the horizontal velocity. In the shallow water approximation $\underline{u}(\hat{r})$ only, independent of z.

If this is true, the linearized shallow water continuity equation has the form

$$\partial_t \xi + \frac{1}{a}\nabla_1 \cdot (h\underline{u}) = 0$$

here $\nabla_1 = \hat{\theta}\,\partial_\theta + \hat{\phi}\,\frac{1}{\sin\theta}\,\partial_\phi$ is the surface grad operator; can write $\nabla_a \equiv \frac{1}{a}\nabla_1$, the surface grad operator on the surface $\Omega_a : r = a$.

We have linearized by neglecting terms of order ξ/h (might break down in shallow water).

In this approximation, the vertical velocity $w(z,\hat{r})$ is a linear function of z.

$$w(z,\hat{r}) = \partial_t \xi \left(\frac{h+z}{h}\right).$$ Then

at $z = 0$, $w(0,\hat{r}) = \partial_t \xi(\hat{r})$

$z = -h$, $w(-h,\hat{r}) = 0$

8. in conjunction with the shallow water or long wave approximation, we shall also assume that the pressure fluctuations are <u>hydrostatic</u>, i.e. at a point $z, \hat{r}$.

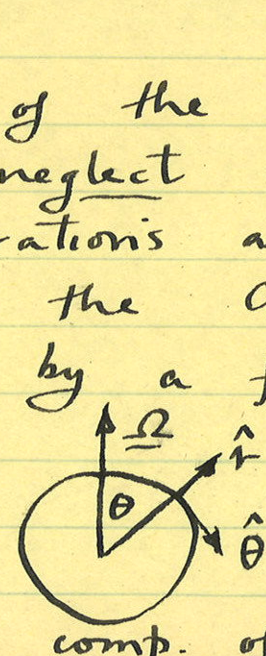
$$p(z,\hat{r}) = (p_a + \rho g z) + \rho g \xi(\hat{r})$$

initial pressure
$p_a \equiv$ atm. pressure

hydrostatic
tidal
fluctuation

Measurements of bottom pressure fluctuations at $\hat{r}$ thus can be converted directly into $\xi(\hat{r},t)$, the principle of pelagic tidal observations.

Adoption of the hydrostatic approximation involves <u>neglect</u> of vertical fluid accelerations and the radial component of the Coriolis acceleration experienced by a fluid particle.



For future reference

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

horizontal comp. of $\underline{\Omega} = \Omega\hat{z}$

hydrostatic approximation involves neglect of $\hat{r} \cdot [\underline{\Omega} \times (\hat{r}w + \underline{u})]$

$$= \hat{r} \cdot \left[\left(\Omega \hat{r} \cos\theta + \Omega \hat{\theta} \sin\theta \right) \times \left(\hat{r}w + \underline{u} \right) \right]$$

$$= \hat{r} \cdot \left[\Omega \hat{\theta} \sin\theta \times \underline{u} \right]$$

Involves only horizontal (or better, <u>tangential</u>) comp. of $\underline{\Omega}$ and tangential velocity $\underline{u}$.

Now consider <u>tan</u>g<u>en</u>tial momentum conservation law, linearized, with hydrostatic approximation. On $r = a$ or $z = 0$:

$$\rho \partial_t \underline{u} + \left\{ 2\rho \underline{\Omega} \times \left[\hat{r}w + \underline{u} \right] \right\}_{\substack{\text{tang.} \\ \text{comp.}}} + \rho g \nabla_a \xi$$

$$= \rho g \nabla_a \bar{\xi}$$

Here the tidal potential driving the tides has been written in terms of the equilibrium tide $\bar{\xi}(\hat{r}, t) = -\frac{1}{g} V(a\hat{r}, t)$.

Consider the Coriolis term, $\underline{\Omega} = \Omega \hat{r} \cos\theta - \Omega \hat{\theta} \sin\theta$

$\hat{r} \times \hat{r} = 0$

$\hat{r} \times \underline{u}$ <u>is</u> tangential

$\hat{\theta} \times \hat{r} = -\hat{\phi}$, tangential

$\hat{\theta} \times \underline{u}$ is radial

Thus get two terms:

$$2\rho\Omega\cos\theta\,\hat{r}\times\underline{u} \;-\; 2\rho\Omega\sin\theta\,w\hat{\phi}$$

9. finally we <u>neglect</u> the term in w

This justified, except maybe near <u>equator</u> $\cos\theta \to 0$, by comparing w and $|\underline{u}|$

$$w = \partial_t \xi\,(h+z)/z \sim \partial_t \xi$$

$$\sim \frac{1}{a}\nabla_1 \cdot (\underline{u}h)$$

$$w/|\underline{u}| \sim h/a \ll 1.$$

Note: 7. 8. 9. are all interdependent. Designated collectively as <u>hydrostatic</u> or <u>traditional</u> <u>approximation</u>.
Note: hydrostatic approximation can be stated as: neglect vertical or radial fluid acceleration and the <u>tangential component</u> of $\underline{\Omega}$. This is what we neglect in 9. also.

Even in the baroclinic situation, Eckart calls this (neglect of $\underline{\Omega}_{tang}$) the traditional approximation. It has been the subject of much historical controversy. The key approximations are 4. (homogeneous fluid) and 7. 8. 9. (traditional) Most recent analysis of these is that of Miles (1974) <u>J.F.M.</u>, <u>66</u>, 241-260,

a very terse paper.
Claims that 4. and 7.8.9. are linked in that $\underline{\Omega}_{\text{tang}}$ can act to couple barotropic and baroclinic modes. This clearly could have profound implications.

We shall restrict attention to the traditional LTE.

$$\partial_t \xi + \nabla_a \cdot (h\underline{u}) = 0$$

$$\rho \partial_t \underline{u} + 2\rho \Omega \cos\theta \, \hat{r} \times \underline{u} + \rho g \nabla_a (\xi - \bar{\xi}) = 0.$$

Note: the main tidal variables $\underline{u}(\hat{r}, t)$, $\xi(\hat{r}, t)$ are independent of z ; z has been eliminated as an independent variable. We shall now define:

$$\left.\begin{array}{l} \underline{k}(\hat{r}) \equiv \Omega \cos\theta \, \hat{r} \\ \nabla \equiv \nabla_a \text{, from now on } \nabla \end{array}\right. , \left\{ \begin{array}{l} \text{we shall not work} \\ \text{in terms of so-called} \\ \text{Coriolis parameter} \\ f \equiv 2\Omega\cos\theta \end{array}\right.$$

$\nabla_a = \frac{1}{a}\left(\hat{\theta}\partial_\theta + \hat{\phi}\frac{1}{\sin\theta}\partial_\phi\right)$ denotes tang. grad. as $\underline{u}$, ξ fcns only of $\hat{r}$ and $\underline{u}$ is a tangent vector.

Also, to cope with tidal friction, introduce
$\underline{F}(\hat{r}) \equiv$ frictional bottom stress [force/area]
$F(\hat{r})$ a tangent vector, $\underline{F}(\hat{r})$ a functional

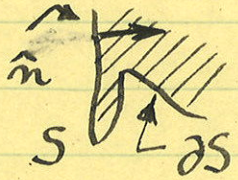
of $\underline{u}(\hat{r},t)$. Then LTE are: Note linear, except for functional $\underline{F}[\underline{u}]$

$$\partial_t \xi + \nabla\cdot(h\underline{u}) = 0$$

$$\star \qquad \partial_t \underline{u} + 2\underline{k}\times\underline{u} + g\nabla(\xi - \bar{\xi}) + \underline{F}/\rho h = 0$$

in oceans $\equiv S$, subset of Ω_a

Subject to b.c. $\hat{n}\cdot\underline{u} = 0$ at coasts $\equiv \partial S$



continent or island

here $\hat{n}$ points out of S, into land areas

i.e. $\hat{n}$ is unit outward normal of S. This is a well-posed problem, when initial values are specified. Note consistent: $\underline{k}\times\underline{u}$ tangent. Conservation of energy may be readily verified. Consider

Note: tidal force $T(\hat{r})$ appears only in tangent vector eqn in hydrostatic approx. $\bar{\xi}(\hat{r})$ arises from need to conserve mass.

Picture of $\underline{T}(\hat{r})$ suggestive of this

$$\int_S \rho h\underline{u}\cdot\underline{\star}\, dA$$

Coriolis forces do no work, as $\underline{u}\cdot(\underline{k}\times\underline{u}) = 0$.

$$\int_S \rho h\underline{u}\cdot\partial_t\underline{u}\, dA = \frac{d}{dt}\int_S \tfrac{1}{2}\rho h\underline{u}\cdot\underline{u}\, dA$$

Consider $\int_S \rho g h\, \underline{u}\cdot\nabla(\xi - \bar{\xi})\, dA$

$$= \int_S \rho g \, \nabla \cdot [\underline{u}(\xi - \bar{\xi})] \, dA - \int_S \rho g (\xi - \bar{\xi}) \nabla \cdot (\underline{u} h) \, dA$$

Gauss theorem on ~~[illegible]~~ curved surfaces

$$= \int_{\partial S} \rho g (\xi - \bar{\xi}) \, \hat{n} \cdot \underline{u} \, dl + \int_S \rho g (\xi - \bar{\xi}) \, \partial_t \xi \, dA$$

zero, by b.c.
no normal velocity
at coasts.

from
continuity

$$= \frac{d}{dt} \int_S \tfrac{1}{2} \rho g \xi^2 \, dA - \int_S \rho g \bar{\xi} \, \partial_t \xi \, dA$$

Summarizing:

$$\frac{d}{dt} [\mathcal{T} + \mathcal{V}] = \int_S \rho g \xi_t \bar{\xi} \, dA + \int_S \underline{u} \cdot \underline{F} \, dA$$

$$\mathcal{T} = \tfrac{1}{2} \int_S \rho h \, \underline{u} \cdot \underline{u} \, dA ,$$ kinetic energy

$$\mathcal{V} = \tfrac{1}{2} \int_S \rho g \xi^2 \, dA ,$$ gravitational potential energy

$\frac{d}{dt} [\mathcal{T} + \mathcal{V}] =$ work done on ocean surface by $\odot$, $☾$ $-$ energy dissipated by tidal friction.

If $\langle\,\rangle$ denotes a long time average, we have

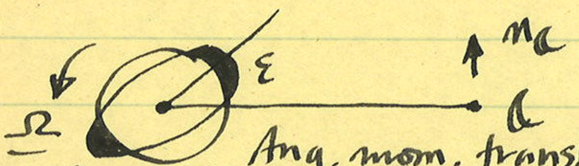
note minus sign

$$\left\langle \int_S \xi_t \xi \rho g \, dA \right\rangle = \left\langle \int_S -\underline{u} \cdot \underline{F}[\underline{u}] \, dA \right.$$

$\langle$work done on ocean by $\odot$, ☾$\rangle$ = $\langle$rate at which energy is dissipated$\rangle$

Friction must oppose the motion, $-\underline{u}\cdot\underline{F} > 0$.

Tidal friction: a brief account. The rate at which tidal energy is being dissipated can be determined from astr. obs. of the motion of the ☾. Simple heuristic picture of tidal friction mechanism (neglects effect of $\odot$ and solar tides & torques)



Ang. mom. transfer from rot. of ⊕ to orbital ang. mom. of ☾.

Phase lag due to dissipation. Let $n_☾ \equiv$ orbital angular velocity of ☾.

The change $\dot{n}_☾$ can be measured by comparing motion of ☾ to ★ or other planets. Motion of ☾ appears non-Keplerian for 2 reasons 1. ⊕ slowing down $\dot{\Omega}$, 2. moon moving out, $\dot{n}_☾$. But ★ and planets affected by $\dot{\Omega}$, not by $\dot{n}_☾$, hence can be measured.

Find $\dot{n}_☾ / n_☾ \sim -2.5 \cdot 10^{-10}$/yr. Moon is moving away from ⊕ as a result

Cons. of $\oplus$☾ angular momentum requires that

$$\frac{d}{dt}\left[C_{\oplus}\Omega_{\oplus} + \frac{m_{\oplus} m_{\leftmoon}}{m_{\oplus}+m_{\leftmoon}}\, a_{\leftmoon}^2 n_{\leftmoon} \right] = 0$$

Note reduced mass. From Kepler's third law $n_{\leftmoon}^2 a_{\leftmoon}^3 = G(m_{\oplus}+m_{\leftmoon})$

Thus

$$\dot{a}_{\leftmoon}/a_{\leftmoon} = -\frac{2}{3}\frac{\dot{n}_{\leftmoon}}{n_{\leftmoon}}, \qquad \dot{a}_{\leftmoon} \sim 6 \text{ cm/yr.}$$

$$[C\dot{\Omega}_{\oplus}]_{\text{tidal}} = \frac{1}{3}\frac{m_{\leftmoon} m_{\oplus}}{m_{\oplus}+m_{\leftmoon}}\, a_{\leftmoon}^2 \dot{n}_{\leftmoon}$$

$$(\dot{\Omega}/\Omega)_{\text{tidal}} \cong -5\cdot 10^{-10} \text{ per year}$$

increase in l.o.d. of ~ 3.5 ms/century

Can be observed, since cumulative.

Can also measure $(\dot{\Omega}/\Omega)_{\text{total}}$; find $(\dot{\Omega}/\Omega)_{\text{non-tidal, but secular}}$ is positive, attributed to isostatic response of ocean floor to increase in $\underline{\xi}$ after melting of last glaciation.

Lunar tidal torque $L_{\leftmoon} = C\dot{\Omega}_{\leftmoon,\text{tidal}}$, and rate at which this torque does work

$$\langle \dot{E} \rangle = L_{\leftmoon}(\Omega - n_{\leftmoon}) \sim 5\cdot 10^{19} \text{ erg/sec}$$

This is the rate at which tidal energy is being dissipated: it is very large, about 30% total heat flow, also about 30% seismic energy release.

This is a constraint on tidal modeling.

$$\langle \dot{E} \rangle = 5\cdot 10^{19} \text{ erg/sec} \equiv -\left\langle \int_S \underline{u}\cdot \underline{F}\, dA \right\rangle$$

$$= \int_S \xi_t \bar{\xi}\, dA.$$

note: this increased recently from $2.7\cdot 10^{19}$ in Munk & MacD.

$\xi(\hat{r}, t)$ must satisfy this constraint.

Note: this is the energy dissipation in ⊕☾ system; can occur in ☾, solid ⊕ or oceans. A simple estimate, based on extrapolation of solid ⊕ Q from 1 hr to 12 hrs, suggests ~ 99% must occur <u>in oceans</u>; the ☾, being smaller & stronger, accounts for even less than solid ⊕. We shall return to the <u>question</u>: where & how is tidal energy dissipated in oceans. Also, would be nice to have a direct check on estimate ~ 1% in solid ⊕.

Now, to gain more insight into LTE, we seek <u>normal mode solutions</u>.

1. no dissipation
2. no driving term $\bar{\xi}(\hat{r}, t)$.
3. $\xi(\hat{r}, t) \sim \xi(\hat{r})e^{i\sigma t}$, $\underline{u}(\hat{r}, t) \sim \underline{u}(\hat{r})e^{i\sigma t}$

Then
$$i\sigma\xi + \nabla\cdot(h\underline{u}) = 0 \quad \text{in } S$$
$$i\sigma\underline{u} + 2\underline{k}\times\underline{u} + g\nabla\xi = 0 \quad \text{in } S$$
$$\hat{n}\cdot\underline{u} = 0 \quad \text{on } \partial S$$

Eliminating ξ by mult. by $i\sigma$

$$\star \quad -\sigma^2 \underline{u} + 2i\sigma \underline{k}\times\underline{u} - g\nabla[\nabla\cdot(h\underline{u})] = 0 \quad \text{in } S$$
$$\hat{n}\cdot\underline{u} = 0 \quad \text{on } \partial S.$$

To be formal; defn: $T \equiv$ space of tangent vectors defined on S which satisfy the b.c. $\hat{n}\cdot\underline{u} = 0$ on ∂S.

$$T \equiv \{u(\hat{r}) : \hat{r}\in S \text{ and } \hat{n}(\hat{r})\cdot\underline{u}(\hat{r}) = 0, \forall \hat{r}\in\partial S\}$$

$\star$ is an eigenvalue problem in T, $\sigma \equiv$ eigenfrequency. Write $\star$ in shorthand

$$\mathcal{L}[\underline{u}] = \underline{0} \quad , \quad \mathcal{L}: T \to T \text{ is linear}$$

defn: inner product on tangent space T

$$(\underline{u}', \underline{u}) \equiv \int_S \rho h\, \underline{u}'\cdot\underline{u}^*\, dA$$

Note $*$: we allow both $\underline{u}(\hat{r})$ and σ to be complex.

Claim that $\mathcal{L}$ is hermitian w.r.t. this inner product, i.e.

Should say: we will now establish formal self-adjointness of normal mode problem in a sense to be discussed. $(u, \mathcal{L}\underline{u}) = (\underline{u}, \mathcal{L}\underline{u})^*$

Proof: consider $\rho h u'\cdot \star^*$ and $\int_S$ result dA; here $u'(\hat{r})$ is some other eigenfunction with a different eigenfrequency σ'. Use Gauss theorem

$$\int_S \rho g h\, \underline{u}'\cdot\nabla\nabla\cdot(\underline{u}^* h) = \underbrace{\int_{\partial S} \rho g h\, (\hat{n}\cdot\underline{u}')\, \nabla\cdot(h\underline{u}^*)\, dl}_{\text{zero}} - \int_S \rho g [\nabla\cdot(h\underline{u}')][\nabla\cdot(h\underline{u}^*)]\, dA.$$

Define <u>bilinear</u> forms

$$\mathcal{T}(\underline{u}',\underline{u}) \equiv \int_S \rho h\, \underline{u}'\cdot\underline{u}^* \, dA \quad ; \text{ kinetic energy}$$

$$\mathcal{W}(\underline{u}',u) \equiv \int_S \rho h\, \underline{u}'\cdot(i\underline{k}\times\underline{u})^* \, dA \quad ; \text{ Coriolis}$$

$$\mathcal{V}(\underline{u}',\underline{u}) \equiv \int_S \rho g\, [\nabla\cdot(\underline{u}'h)][\nabla\cdot(\underline{u}^*h)]\, dA \quad ;$$

grav. potential energy

Find

$$\star \quad -\overset{*}{\sigma}{}^2\, \mathcal{T}(\underline{u}',\underline{u}) + 2\sigma^*\mathcal{W}(\underline{u}',\underline{u}) + \mathcal{V}(\underline{u}',\underline{u}) = 0$$

Easily verified that all of $\mathcal{T},\mathcal{W},\mathcal{V}$ are <u>hermitian</u> <u>bilinear</u> <u>forms</u>: $\mathcal{T},\mathcal{V}$ by inspection

$$\mathcal{T}(\underline{u}',\underline{u}) = \mathcal{T}^*(\underline{u},\underline{u}')$$

$$\mathcal{V}(\underline{u}',\underline{u}) = \mathcal{V}^*(\underline{u},\underline{u}')$$

But $\mathcal{W}$ is also, since

$$\mathcal{W}(\underline{u}',\underline{u}) = -i\int_S \rho h\, \underline{u}'\cdot k\times\underline{u}^*\, dA$$

$$= i\int_S \rho h\, \underline{u}^*\cdot\underline{k}\times\underline{u}\, dA$$

$$= \left[\int_S \rho h\underline{u}\cdot(i\underline{k}\times\underline{u}')^*\right]^* = \mathcal{W}^*(\underline{u},\underline{u}').$$

Reversing roles of primed and unprimed eigensolutions yields

$$\star\star \quad -\overset{*}{\sigma}{}'^2\, \mathcal{T}(\underline{u},\underline{u}') + 2\sigma'\mathcal{W}(\underline{u},\underline{u}') + \mathcal{V}(\underline{u},\underline{u}') = 0$$

Now all of $\mathcal{T}(\underline{u},\underline{u})$, $\mathcal{W}(\underline{u},\underline{u})$, $\mathcal{V}(\underline{u},\underline{u})$ are <u>real</u>, because of Hermitian symmetry. Thus, setting $\underline{u} = \underline{u}'$ in either ★ or ★★ and taking ✱

$$\sigma^2 \mathcal{T}(\underline{u},\underline{u}) - 2\sigma \mathcal{W}(\underline{u},\underline{u}) - \mathcal{V}(\underline{u},\underline{u}) = 0$$

Consider as a <u>quadratic equation</u> for σ, if $\underline{u}(\hat{r})$ is known. Solve for σ

$$\sigma = \frac{1}{\mathcal{T}}\left[\mathcal{W} \pm \left(\mathcal{W}^2 + \mathcal{T}\mathcal{V}\right)^{1/2}\right]$$

Since $\mathcal{V}(\underline{u},\underline{u})$ is <u>clearly</u> <u>positive definite</u>, every eigenfrequency σ <u>must be real</u>.

Note: it is easily demonstrated that $\mathcal{V}$ positive definite is a <u>necessary</u>, as well as sufficient condition for <u>secular stability</u> of the oceans.

Now in ★ and ★★, take σ and σ' to be real; appeal to symmetry of $\mathcal{V}$, $\mathcal{T}$, $\mathcal{W}$. Find an <u>orthogonality</u> (<u>quasi</u>-orthogonality) relation.

$$\mathcal{T}(\underline{u}',\underline{u}) - \frac{2}{\sigma' + \sigma}\mathcal{W}(\underline{u}',\underline{u}) = 0$$

<u>True</u> orthogonality $(\underline{s}',\underline{s}) = \mathcal{T}(\underline{s}',\underline{s})$ doesn't <u>prevail</u>

This not quite right, mostly because our shorthand notation is too short, i.e. $\mathcal{L}: T \to T$ as defined contains the eigenfrequency σ. Thus $(\underline{u}', \mathcal{L}\underline{u}) \neq (\underline{u}, \mathcal{L}\underline{u}')^*$.

The hermitian symmetry of $\mathcal{I}, \mathcal{W}, \mathcal{V}$ implies immediately that $\mathcal{L}: T \to T$ is a hermitian linear operator. Note: T must be a complex space.

Hermitian symmetry gives rise to a useful variational principle (generalized Rayleigh's principle). Namely

Consider $(\underline{u}, \mathcal{L}\underline{u}) \equiv \mathcal{L}(\underline{u}, \underline{u})$

$$\equiv \sigma^2 \mathcal{I}(\underline{u}, \underline{u}) - 2\sigma \mathcal{W}(\underline{u}, \underline{u}) - \mathcal{V}(\underline{u}, \underline{u})$$

as a (bilinear) functional on T.
Then $\mathcal{L}(\underline{u}, \underline{u})$ is stationary i.e. $\delta\mathcal{L}(\underline{u}, \underline{u}) = 0$ iff $\underline{u}(\hat{r})$ is an eigenfunction with eigenfrequency σ.

Proof: $\delta\mathcal{L}(\underline{u}, \underline{u}) = \delta(\underline{u}, \mathcal{L}\underline{u})$

$$= (\delta\underline{u}, \mathcal{L}\underline{u}) + (\underline{u}, \mathcal{L}\,\delta\underline{u})$$

$$= (\delta\underline{u}, \mathcal{L}\underline{u}) + (\delta\underline{u}, \mathcal{L}\underline{u})^*, \quad \underline{\text{Hermitian}}$$

$$= 2\,\mathrm{Re}\,(\delta\underline{u}, \mathcal{L}\underline{u}) = 0 \quad \text{for arbitrary small variations } \delta\underline{u} \text{ iff } \mathcal{L}\underline{u} = \underline{0}.$$

To my knowledge, little practical advantage has been made of this principle. It opens the door to Rayleigh-Ritz methods for finding ocean basin normal modes.

To gain familiarity with LTE, we now seek normal mode solutions for the simplest case, A uniform covering ocean of depth h.

Hough 1898 Phil Trans A 191, 139–185

Longuet-Higgins Phil Trans A 262, 511–607

Case 1: <u>no rotation</u> $\Omega \equiv 0$

$$i\sigma\xi + h\,\nabla\cdot\underline{u} = 0 \qquad \text{remember} \quad \nabla \equiv \frac{1}{a}\nabla_1$$
$$i\sigma\underline{u} + g\nabla\xi = 0 \qquad\qquad \nabla\cdot\underline{u} \equiv \frac{1}{a}\,\mathrm{tr}(\nabla_1\underline{u})$$

We use the <u>tangent vector</u> representation (useful as well in general case)

$$\underline{u}(\hat{r}) = \underbrace{\nabla_1 V(\hat{r})}_{\text{poloidal part}} \underbrace{- \hat{r}\times\nabla_1 W(\hat{r})}_{\text{toroidal part}}$$

$\xi(\hat{r}), V(\hat{r}), W(\hat{r})$ scalar fields on S. Can be expanded in $Y_\ell^m(\hat{r})$. Call the coefficients $\xi_\ell^m, V_\ell^m, W_\ell^m$.

Then
$$\underline{u}(\hat{r}) = \sum_{\ell m} V_\ell^m\,\nabla_1 Y_\ell^m(\hat{r}) + W_\ell^m\left[-\hat{r}\times\nabla_1 Y_\ell^m(\hat{r})\right]$$

These are, except for a normalization factor, the <u>vector</u> spherical harmonics $\underline{B}_\ell^m(\hat{r}), \underline{C}_\ell^m(\hat{r})$ defined e.g. by Morse & Feshbach, vol II.

They are mutually orthogonal in the sense

$$\int_\Omega \underline{B}_\ell^m \cdot \overset{*}{\underline{B}}{}_{\ell'}^{m'} \, dA = \ell(\ell+1)\, \delta_{\ell\ell'}\, \delta_{mm'}$$

$$\int_\Omega \underline{C}_\ell^m \cdot \overset{*}{\underline{C}}{}_{\ell'}^{m'} \, dA = \ell(\ell+1)\, \delta_{\ell\ell'}\, \delta_{mm'}$$

$$\int_\Omega \underline{B}_\ell^m \cdot \overset{*}{\underline{C}}{}_{\ell'}^{m'} \, dA = 0.$$

Note $\nabla\cdot\underline{u} \equiv \frac{1}{a}\nabla_1\cdot\underline{u} = \frac{1}{a}\nabla_1^2 V$; toroidal part has no surface divergence

and $\nabla_1^2 Y_\ell^m = -\ell(\ell+1) Y_\ell^m$

$$\nabla\cdot\underline{u} = \sum_{\ell m} -\frac{\ell(\ell+1)}{a} V_\ell^m Y_\ell^m$$

Consider also $\hat{r}\cdot\nabla\times\underline{u}$. Note: really should consider the Cartan exterior derivative $\underline{du}$. To deduce $\hat{r}\cdot\nabla\times\underline{u}$, need to extend $\underline{u}(\hat{r})$ away from $r=a$; answer however doesn't depend on how this extension is done, since

$$\hat{r}\cdot\nabla\times\underline{u} = -\frac{1}{a}\nabla_1^2 W$$

$$= \sum_{\ell m} \frac{\ell(\ell+1)}{a} W_\ell^m Y_\ell^m ,$$

poloidal part has no vorticity, but toroidal part does.

We thus get

$$\sum_{\ell m} \left[i\sigma \xi_\ell^m - \frac{h}{a} \ell(\ell+1) V_\ell^m \right] Y_\ell^m = 0$$

$$\sum_{\ell m} \left[i\sigma V_\ell^m + \frac{g}{a} \xi_\ell^m \right] \nabla_1 Y_\ell^m - \left[i\sigma W_\ell^m \right] \hat{r} \times \nabla_1 Y_\ell^m = 0$$

Two types of modes:

1. secular or steady <u>toroidal</u> <u>flows</u>
   $\sigma = 0$
   W_ℓ^m arbitrary , $\xi_\ell^m = 0$; no radial displacement
   <u>any</u> toroidal flow (these have vorticity) is a steady solution

2. gravity modes
   $W_\ell^m = 0$; no toroidal part

   $$V_\ell^m = \frac{i\sigma a}{\ell(\ell+1) h} \xi_\ell^m , \quad \text{out of phase}$$

   $$\sigma^2 = \ell(\ell+1) \frac{gh}{a^2} \quad ; \quad 2\ell+1 \text{ degeneracy,}$$
   to be expected because of spherical symmetry.

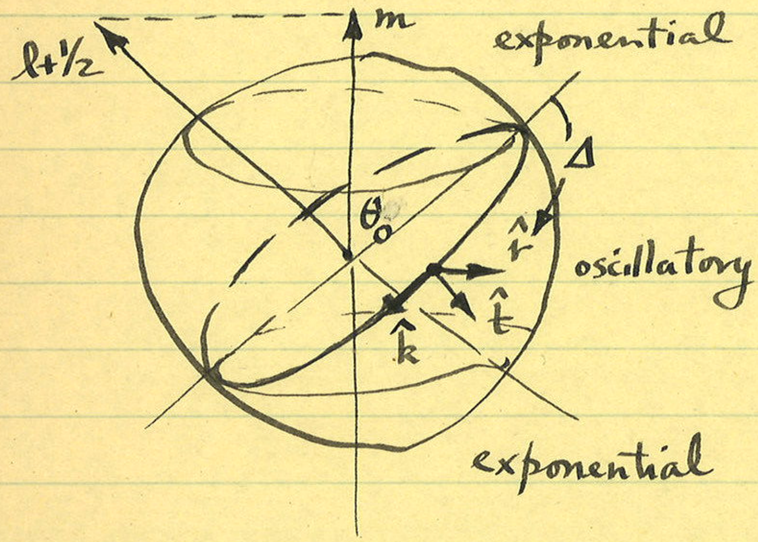
Note: in absence of Ω, a mode is purely poloidal or purely toroidal and char. by a <u>single</u> $Y_\ell^m(\hat{r})$. This makes sense.

Thus $V_\ell^m = i \sqrt{\dfrac{g}{\ell(\ell+1) h}} \, \xi_\ell^m$

Now a reminder: <u>asymptotic</u> <u>form</u> of $Y_\ell^m(\hat{r})$ for large degree ℓ; see Edmonds 1960.

Every Y_ℓ^m has two turning colatitudes θ_0 and $\pi-\theta_0$

$$\theta_0 \equiv \sin^{-1}\left|\frac{m}{\ell+1/2}\right|$$



Assoc. with every Y_ℓ^m: family of precessing great circles, all tangent to turning colatitudes

In the oscillatory mid-latitude region

$$Y_\ell^m(\hat{r}) = \frac{1}{2\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4} e^{i\left[k\Delta + N\frac{\pi}{2} - \frac{\pi}{4}\right]} + O\left(\frac{1}{\ell+\frac{1}{2}}\right)$$

$$k \equiv \frac{\ell+1/2}{a}$$

Note $\Delta(\hat{r})$ well defined. $\forall \hat{r}$, only one great circle through. $N=0$ front face, $N=1$ back face. Note $\sqrt{\ell(\ell+1)} = \ell + 1/2 + O(1/\ell+1/2)$

The complex eigenfunctions $\xi_\ell^m Y_\ell^m(\hat{r})$ thus look like propagating surface gravity waves

$$\xi(\hat{r},t) \sim \frac{\xi}{2\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4} e^{i\left[k\Delta + N\frac{\pi}{2} - \frac{\pi}{4} \pm \sigma t\right]}$$

with wavenumber k. The associated currents are <u>along</u> the direction of propagation

$$u(\hat{r},t) \sim -\hat{k}\sqrt{g/h}\;\xi\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4} e^{i\left[k\Delta + N\frac{\pi}{2} - \frac{\pi}{4} \pm \sigma t\right]}$$

this is the appropriate shallow water factor

N increases by 1 at every <u>caustic</u> encounter

$$\sigma^2 = gh k^2 \qquad \sigma/k = c = \sqrt{gh} \text{ , phase speed.}$$

These are familiar shallow water surface gravity waves propagating around the ocean on great circle paths.
Clear that since phase speed is independent of direction (Y_ℓ^m and Y_ℓ^{-m} have same σ), can compose <u>real</u> standing wave normal modes.
This <u>not</u> true with Ω ; then all modes must be complex and propagating.

Case II : <u>slow rotation</u> $2a\Omega \ll \sqrt{gh}$
Actually in deep ocean basins $h \sim 4$ km
$2a\Omega / \sqrt{gh} \sim 0.4 = O(1)$.

We can use Rayleigh's principle to find the first order correction to the eigenfrequencies. Consider first the surface gravity modes. In the presence of slow rotation, $\forall$ mode

$$\sigma^2 \mathcal{J}(\underline{u},\underline{u}) - 2\sigma \mathcal{W}(\underline{u},\underline{u}) - \mathcal{V}(\underline{u},\underline{u}) = 0$$
$$\mathcal{L}(\underline{u},\underline{u}) = 0$$

We know 1. $\mathcal{L}(\underline{u},\underline{u})$ is <u>stationary</u>
2. $\mathcal{L}(\underline{u},\underline{u}) = 0$

Treat as a functional of σ and $\underline{\Omega}$ as well as $\underline{u}$. Consider a variation in $\underline{\Omega}$ away from zero.

zero, by Rayleigh

$$2\sigma\,\delta\sigma\, J - 2\sigma W + \cancel{\left[\sigma^2 \delta J - \delta V\right]} = 0$$

$$\delta\sigma = \frac{W(\underline{u},\underline{u})}{J(\underline{u},\underline{u})}$$

. This integral easily evaluated for $\underline{u} = V_\ell^m \nabla_1 Y_\ell^m$

Note: actually must deal with degeneracy but if $\underline{\Omega} = \Omega\hat{z}$, "$m$ is a good quantum number". This the exact analogue of Zeeman splitting of H atom. Find

$$\delta\sigma = \frac{m\Omega}{\ell(\ell+1)}$$

, splitting is symmetric à la Zeeman

This is first order perturbation theory

Gravity waves going with the $\oplus$ go more slowly than those going ~~[illegible]~~ against. This effect is maximum for $m = \pm \ell$, waves going around equator. For $\ell \gg 1$, $\ell + 1/2 \sim ka$, for such equatorial waves

$$c = \sqrt{gh}\left[1 \pm \frac{\Omega a}{\sqrt{gh}}(ka)^{-2}\right]$$

$\frac{\lambda}{2\pi a}$ very small for wind-generated gravity waves

This a very small effect for wind waves, obviously

More important is the first order effect of slow rotation on the eigenfunctions. We shall <u>not</u> discuss this <u>quantitatively</u>. Can be shown that:

$$\underline{u} = \underline{\sigma}_\ell^m + \underline{\tau}_{\ell\pm1}^m$$

Because of this, the eigenfns of the gravity waves will have vorticity.

poloidal degree ℓ — toroidal, degrees $\ell\pm1$ <u>first order</u> correction

Physically, Coriolis force will act to deflect particle motion, give rise to counter-clockwise amphidromes in N hemisphere and clockwise in S. This is certainly the tendency in cotidal charts.

Now, more interesting, effect of slow $\underline{\Omega}$ on steady toroidal flow modes. For these, on a non-rotating $\oplus$, $\xi(\hat{r}) = 0$, so $\mathcal{V}(\underline{u},\underline{u}) = 0$, and on a rotating $\oplus$

$$\underbrace{\sigma\,\delta\mathcal{J}(\underline{u},\underline{u})}_{\text{zero, by Rayleigh}} + \delta\sigma\,\mathcal{J}(\underline{u},\underline{u}) - 2\mathcal{W}(\underline{u},\underline{u})$$

$$\delta\sigma = \frac{2\mathcal{W}(\underline{u},\underline{u})}{\mathcal{J}(\underline{u},\underline{u})}, \quad \underline{u} = W_\ell^m\left[-\hat{r}\times\nabla_1 Y_\ell^m\right]$$

Find $\delta\sigma = \sigma$, to first order in $\frac{2\Omega a}{\sqrt{gh}}$

$$\sigma = \frac{2m\Omega}{\ell(\ell+1)}$$

note: correct to first order, σ independent of h.
For large wavenumber $l + \frac{1}{2} \gg 1$.
Useful to define β

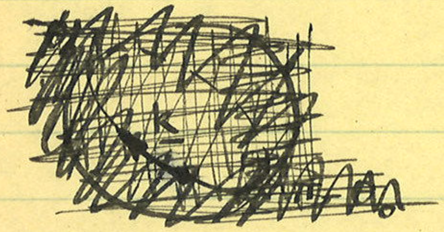
$$\beta \equiv -\frac{1}{a}\partial_\theta \left[2\Omega \cos\theta \right] \qquad \text{, corresponds to}$$
$$= \frac{2\Omega}{a}\sin\theta \qquad\qquad \beta \equiv \partial f/\partial y \text{ of}$$

β-plane

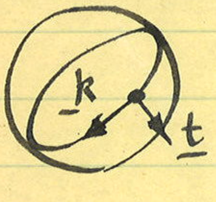
For $ka = l + \frac{1}{2} \gg 1$.
Phase speed is $c = \frac{\sigma}{k} = \frac{2m\Omega}{(ka)^2}\left(\frac{1}{k}\right)$

$$\sigma = \frac{2\Omega \sin\theta}{a} \; \frac{m}{a\sin\theta} \; \frac{1}{k^2} = \beta \frac{m}{a\sin\theta}\frac{1}{k^2}$$

Now $\frac{m}{a\sin\theta} \equiv k_\phi$, ϕ-component of $\underline{k} \equiv$ wave number vector.



at caustic colatitudes $\theta_0, \pi - \theta_0$,
$\sin\theta_0 = \frac{|m|}{l + 1/2}$, $k_\phi = k$.



wave number vector
$$\underline{k} = \left(\pm \frac{1}{a} p_\theta, \frac{m}{a\sin\theta}\right)$$
where $p_\theta^2 = (l + 1/2)^2 - m^2/\sin^2\theta$.

dispersion relation $\sigma = \beta k_\phi / k^2$

Phase speed: $c = \beta k_\phi / k^3$, familiar phase speed of Rossby or planetary waves on an equatorial β plane.
Using the asymptotic expression for Y_l^m:

$$u(\hat{r}, t) \sim -ika\,\hat{t}\,W \frac{1}{2\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4} e^{i\left[k\Delta + N\frac{\pi}{2} - \frac{\pi}{4} + \sigma t\right]}$$

Currents are transverse to the direction of propagation. Note: for any mode, σ is single-signed.
Rossby waves can propagate only from E to W.
To see this simply: $\sigma = \frac{2m\Omega}{\ell(\ell+1)}$ $\quad Y_\ell^m e^{i\sigma t}$

$$\sim \exp i\left[m\phi + 2m\Omega/\ell(\ell+1) \right]$$

Both $\pm m$ propagate E to W.

This division into two types of modes due to Laplace: his designation first and second types (latter $\equiv$ Rossby).

The normal mode spectrum of even a uniform ocean is thus enormously rich.

1. gravity modes, with arbitrarily short periods, as well as $T \sim 12$ hrs (time to cross an ocean basin)
2. Rossby modes, with arbitrarily long periods, as well as periods $T \sim 12$ hr.
3. variable bottom relief introduces additional types of topographic modes. These are not readily studied in the global context we have adopted here.

and this is for a uniform ocean.

The existence of the Rossby modes invalidates any simple-minded proof that M_f and other long-period tides will be equilibrium. Wunsch's analyses demonstrate convincingly that they are not.

At this point tidal <u>theory</u> bifurcates:

1. <u>local</u>: f-plane, β-plane, with topography, sloping bottoms, might as well add stratification, etc. Such studies almost always make the <u>traditional</u> approximation $\underline{\Omega}_{horiz} \equiv \underline{0}$. One wonders about this.
2. numerical attacks upon the <u>global</u> problem. <u>We discuss this briefly</u>. These are <u>finite</u> <u>difference</u> <u>calc</u>.

In every such study one <u>assumes</u> $\xi(\hat{r},t) \sim \xi(\hat{r})e^{i\sigma t}$, $\underline{u}(\hat{r},t) \sim \underline{u}(\hat{r})e^{i\sigma t}$, σ = the frequency of a major tidal constituent, generally only M_2. One takes $\bar{\xi}(\hat{r},t) = M_2$ tide only, in that case. The big decision: how to model <u>dissipation</u>. A model which allows for dissipation in detail is out of the question.

The two main ploys are represented by the computations of:

1. Pekeris and Accad, Phil Trans A, $\underline{265}$, 413–436, 1969: introduce an artificial bottom friction, <u>linear</u> in $\underline{u}(\hat{r})$ "for analytical convenience".

$$\underline{F} \propto -\underline{u}/h^n \quad , \quad \text{take } n = 3$$

to limit dissipation to near coastal regions
Reason for this: dissipation is widely believed to be concentrated in <u>shallow</u> marginal basins.
An empirical "law" for turbulent bottom friction $\underline{F} = -\kappa\rho|\underline{u}|\underline{u}$, $\kappa \sim .002-.003$
Pekeris and Accad use as b.c. at coasts, $\hat{n}\cdot\underline{u} = 0$. They have done global M_2 tide: seek to compare with observed coastal and island obs. Output is a cotidal and corange chart for world's oceans, M_2 tide

2. Hendershott, Ann. Rev. Fl. Mech. $\underline{2}$, 205–224, 1970
Dissipationless, $\underline{F}[\underline{u}] = \underline{0}$. Use as b.c. at coasts <u>observed tidal hts</u> $\xi(\hat{r})$ extrapolated where necessary, e.g. Antarctica.
Easy to demonstrate this also a well-posed problem, i.e. <u>unique answer</u>.
Given a harmonic tidal potential $\Xi(\hat{r})e^{i\sigma t}$, suppose $\exists$ two solutions $\xi', \underline{u}'$ and $\xi'', \underline{u}''$.
Let $\xi = \xi' - \xi''$, $\underline{u} = \underline{u}' - \underline{u}''$. Then $\underline{u}, \xi$ satisfy

comparison then made with <u>island</u> observations.

$$\star \quad i\sigma \underline{u} + 2k \times \underline{u} + g\nabla\xi = 0 \leftarrow \text{by subtraction}$$
$$i\sigma\xi + \nabla\cdot(h\underline{u}) = 0$$

Take $\rho h \underline{u} \cdot \star$ and $\int_S dA$, Use Gauss' theorem

$$i\sigma \int_S \rho h \underline{u}\cdot\underline{u}\, dA + i\sigma \int_S \rho g \xi^2 dA$$
$$+ \int_{\partial S} \rho g h \xi\, \hat{n}\cdot\underline{u}\, dA.$$

Now if either $\xi = 0$ (i.e. both ξ', ξ'' equal) or $\hat{n}\cdot\underline{u} = 0$ (i.e. both $\hat{n}\cdot\underline{u}'$, $\hat{n}\cdot\underline{u}''$ equal) on ∂S, we have

$$i\sigma \left[\int_S [\rho h |\underline{u}|^2 + \rho g h \xi^2 \right] dA = 0.$$

Sum of two positive terms in integrand. Hence both $|\underline{u}|$ and ξ must be $\equiv 0 \quad \forall \hat{r} \in S$.
q.e.d. Note: σ must be $\neq 0$.

Note: implicit in prescription $\xi(\hat{r})$ = obs. values on ∂S is that $\hat{n}\cdot\underline{u}$ might be non-zero. Flow through computational boundaries. This interpreted as near shore dissipation (power flows into beaches where dissipated. This is, to certain extent, inconsistent with other assumption: that observed coastal tides are representative of adjoining deep water values.

There is talk of using <u>local</u> non-linear theories to extrapolate seaward for "representative" boundary values for the global problem.

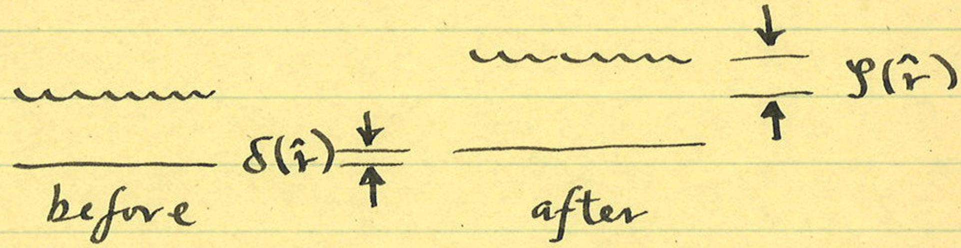
Comparison of M_2 results. Quote from Hendershott "... the global models are disturbingly sensitive to small perturbations, the S. Atlantic amphidrome comes and goes, Indian Ocean amplitudes fluctuate by factors of two or three, Pacific amphidromes move about, now nearly coalescing, and now rotating in the wrong direction ..."

This almost certainly due to near resonance with one or several normal modes (not surprising in view of the demonstrated diversity of modes).

Left out of <u>all</u> global models to date: effects of solid ⊕ deformation in response to tidal loading and gravitational self attraction of the oceans. Only recently appreciated that this is an extremely important effect; might be responsible for disagreement between global models and observations, although it is also evident that <u>friction</u> has not yet been properly parameterized.

Inclusion of solid Earth deformation and oceanic gravitational self-attraction.

Let



$\varsigma(\hat{r}) \equiv$ geocentric \$ elevation

$\delta(\hat{r}) \equiv$ geocentric ocean bottom elevation

The ocean tide is then $\xi(\hat{r}) = \varsigma(\hat{r}) - \delta(\hat{r})$

Continuity equation, in shallow water approx. is still.

$$\partial_t \xi + \frac{1}{a} \nabla_1 \cdot (h \underline{u}) = 0 \qquad \text{or}$$

$$\partial_t (\varsigma - \delta) + \frac{1}{a} \nabla_1 \cdot (h \underline{u}) = 0$$

Tangential momentum conservation law is

$$\partial_t \underline{u} + 2 \underline{k} \times \underline{u} + g \nabla (\varsigma - \bar{\xi}) + \underline{F}/\rho h = 0$$

note

where $\bar{\xi}(\hat{r}, t) = -\frac{1}{g} \Big[\sum_\ell (1 + k_\ell - h_\ell) V_\ell(\hat{r}, t)$

$+ \tilde{V}(\hat{r}, t) \Big]$

where $\tilde{V}(\hat{r}, t) \equiv$ grav. pot. due to oceanic self-attraction + yielding of solid $\oplus$ to tidal load.

$\tilde{V}(\hat{r},t)$ has two distinct terms; reduces to zero only if $\rho_w = 0$. Even in that case $\bar{\xi}(\hat{r},t)$ differs from that on a rigid ⊕ by factor $1+k-h$, taking $V_\ell \approx 0$, $\ell > 2$. This due to defm of solid ⊕ caused by direct influence of tidal potential.

Note: Pekeris even neglects this term in his global comp., although since he does a linear problem, even linear friction, he can multiply every tidal ht. $\xi(\hat{r})$ by $1+k-h \approx .691$. In his paper however, he does not.

Hendershott computed M_2 using $g\bar{\xi} = (1+k-h)V_2$ i.e. $\tilde{V}(\hat{r},t) = 0$. Farrell, Phil. Trans. A, 1972, using a Green's function for a realistic ⊕ model, then computed $\tilde{V}(\hat{r},t)$ for his computed M_2 tide. In principle, this the first step in a self-consistent iterative solution.
Comp. done using load Love numbers, which no time to discuss.
Surprising result: magnitude of $\tilde{V}(a\hat{r},t)$ compared to $[1+k-h]V(a\hat{r},t) \propto P_2^2(\cos\theta)$
Two of equal magnitude, $\tilde{V}(a\hat{r})$ esp. large in Indian Ocean, where Hendershott M_2 tide was large. (these actually unrealistically high).

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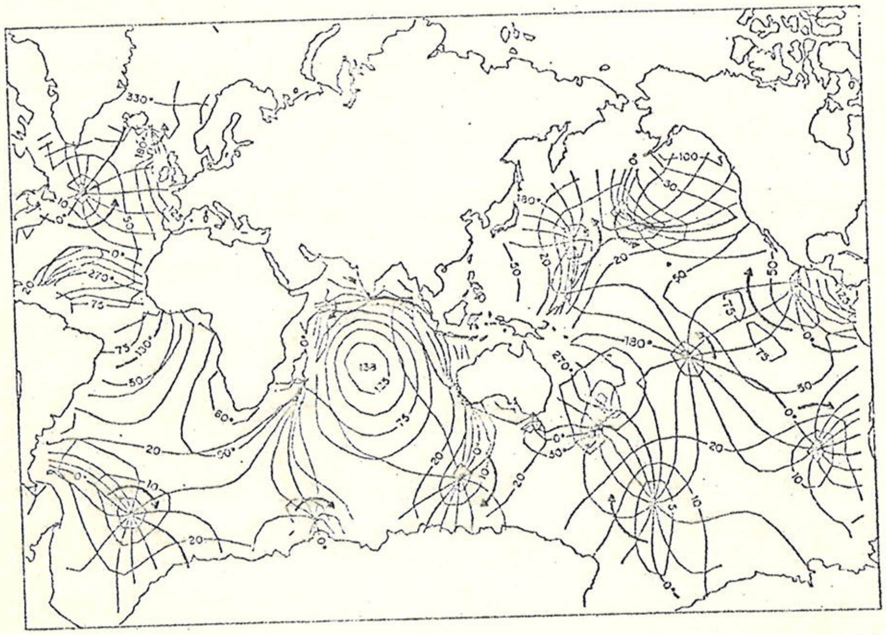

FIG. 2. Cotidal and corange lines for the M_2 tide obtained by solving LTE with coastal values specified. Cotidal lines radiate from amphidromic points, corresponding to the progression of tidal crests in the sense indicated by the heavy arrows around these points of vanishing tidal range (Section 5). High tide occurs along the cotidal lines labeled 0° just as the moon passes over Greenwich meridian. Successive cotidal lines delineate tidal crests at lunar hourly intervals. (For clarity, only selected cotidal lines are labeled, 30° corresponds to a delay of one lunar hour.) Corange lines (5, 10, 20, 50, 75, 100, 125 cm) connect locations of equal tidal amplitude (not double amplitude). They surround amphidromic points (where range vanishes and phases vary rapidly) and range maxima (where range is largest and phases nearly constant).

Semi-empirical charts.—A numerical treatment of any realistic model is mandatory. Defant (62, 63) attempted to use Green's (64) law of propagation for long waves in a canal of slowly varying cross section to describe diurnal and semidiurnal tides in the Atlantic and in marginal seas. Recently, Nekrasov (65) has given a similar discussion of tides in the Arctic Sea. The principle defect in these essentially one-dimensional computations is the neglect of Coriolis forces; the neglect was remedied in part by working with plane rectangular basins (section 5) whose size and orientation were chosen to model small seas crudely. Kelvin and Poincaré waves could then be included explicitly in the solutions (Godin 66, Defant 61).

Numerical schemes.—With the advent of automatic methods of computation, application of the method of finite differences to the tidal problem was inevitable. Hansen's (67) calculation of the M_2 tide in the North Atlantic was the precursor of similar studies by Accad & Pekeris (68); Bogdanov,

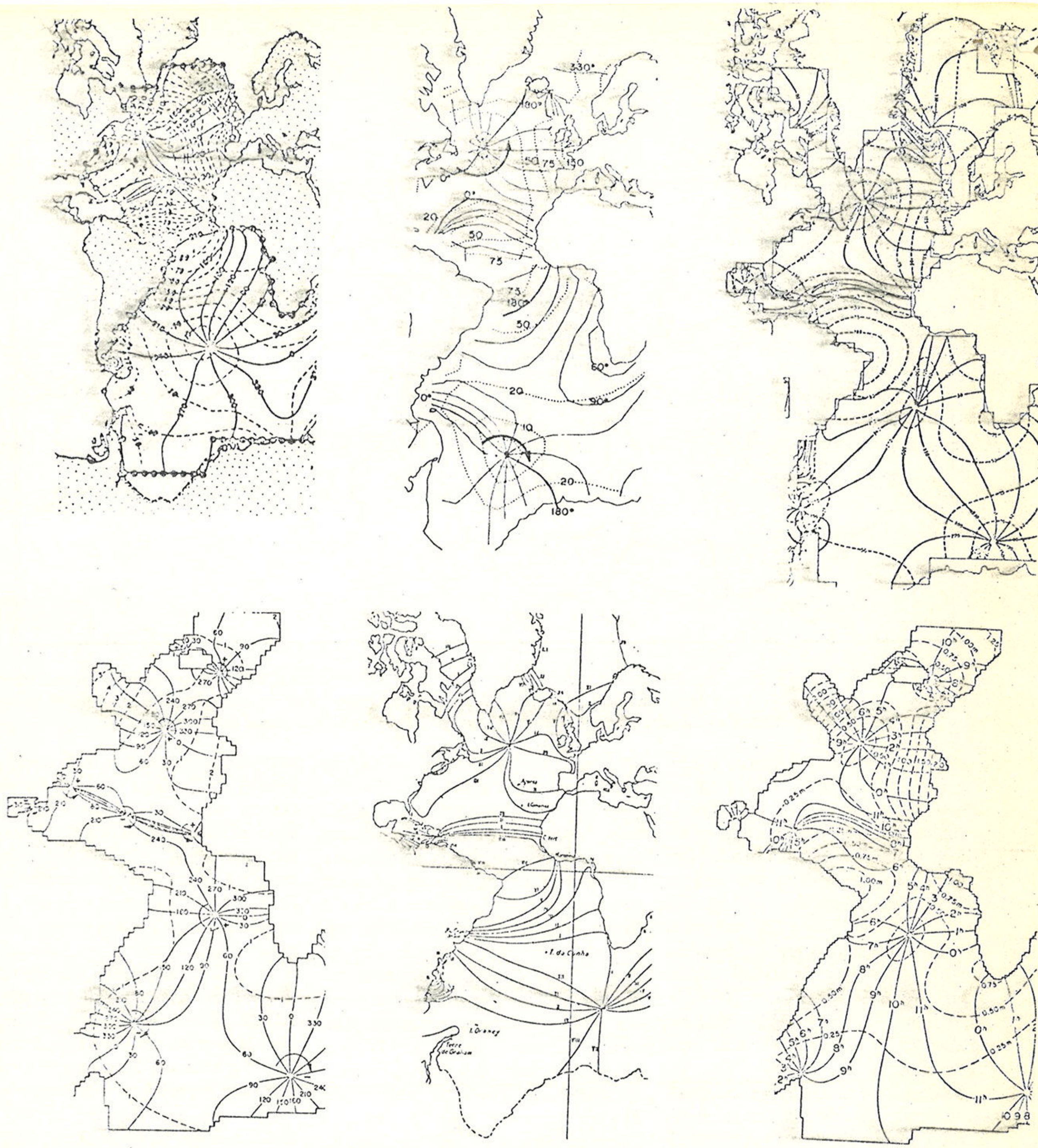

Fig. 3. Atlantic results of global numerical solutions of Laplace's tidal equations for the M2 tide by (left to right, top to bottom) Bogdanov and Magarik [1967], Hendershott [1972], Zahel [1970], Pekeris and Accad [1969],–frictionless, 2° mesh, Dietrich [1944],–an empirical map rather than a numerical solution, redrawn by Villain [1962], Pekeris and Accad [1969],–dissipative, 1° mesh.

feature discovered by the numerical tidalists.

Islands in the Indian Ocean are sufficiently frequent and evenly distributed that a rather good empirical map may be constructed (Figure 4). The salient feature of such a map is a very large region in the south central Indian Ocean within which the sea surface moves nearly synchronously. Every numerical map produces such an 'antiamphidrome.' The computations of *Pekeris and Accad* [1969] and *Hendershott* [1972] suggest that this region is nearly resonant at semidiurnal periods. Hendershott (unpublished) finds that by varying the mean depth by several hundred meters, computed *M*2 amplitudes may be changed by a multiplicative factor of 2 or 3. A very realistic prediction of Indian Ocean island semidiurnal tides may be made by proper empirical choice of the mean depth.

In the Pacific, the various numerical and normal mode models show the greatest divergence from one another and from Pacific island observations (Figure 5). This is perhaps not surprising in view of the size of this basin and the consequent diminution of the direct influence of coastal boundary conditions on mid-basin tides.

The empirical cotidal maps show a

Effect of $\tilde{V}(a\hat{r},t)$ may even exceed that of astronomical potential since the excitation amplitude of the nearly resonant ocean basin normal modes must be, roughly (this not exact because of Ω)

$$\text{amplitude}_n \sim (\bar{\xi}, \underset{\uparrow}{\xi_n})$$

eigenfunction of n^{th} mode

The eigenfunctions with $T \sim 12$ hrs will have scale lengths $\sim$ ocean basin, $\tilde{V}$ has $\sim$ same, but $[1+k-h]V$ is $\propto P_2^2(\cos\theta)$

Hendershott (1972) G.J.R.A.S. 29, 389–402 has computed the first iterate for M_2 adding in Farrell's $\tilde{V}(a\hat{r},t)$ from the zeroeth iterate. Finds, e.g. that his unrealistic Indian Ocean amplitudes are reduced.

The relative infancy of these studies is clear: where else do people publish their iterations, no discussion of convergence.

My proposed research: a $Y_l^m(\hat{r})$ approach; compare with satellite gravimetry and altimetry.