18. Spherical harmonics

Investigation of the free oscillations of a spherical ⊕ is aided considerably by a knowledge of spherical harmonics. For this reason, we shall take a few lectures for a mathematical aside on these objects. We shall follow some notes of George Backus. A useful reference is his paper in Annals of Physics 4, 372-447 (1958), written while he was at Princeton.

Sect A.7 Spherical polar coordinates

First a few definitions.

Ω : surface of unit sphere centered on origin $\underline{O}$ in $\mathbb{R}^3$.

$\mathcal{L}_2(\Omega)$: all complex valued functions $f : \Omega \to \mathbb{C}$ s.t.

$$\int_\Omega f f^* \, dA < \infty,$$ i.e. square integrable

inner product on $\mathcal{L}_2(\Omega)$:

$$(f, g) = \int_\Omega f g^* \, dA$$

Then $\mathcal{L}_2(\Omega)$ is a Hilbert space. $\dim \mathcal{L}_2(\Omega) = \infty$

We wish to examine two <u>subspaces</u> of $\mathcal{L}_2(\Omega)$.

<u>Def</u>: a polynomial in R_3 with complex coefficients is called a <u>homogeneous polynomial of degree l</u> if it involves terms only of total degree l, i.e.

in DFT

we call this $\mathcal{H}_l(\mathbf{r})$

$$P_l(\underline{r}) = \sum C_{\alpha\beta\gamma}\, x^\alpha y^\beta z^\gamma \quad ; \quad \alpha+\beta+\gamma = l$$

give some examples

Note: if $P_l(\hat{r})$ is known and $P_l(\underline{r})$ is homogeneous, then $P_l(\underline{r})$ is determined for all $\underline{r}$, since

$$P_l(\underline{r}) = r^l P_l(\hat{r})$$

or $$P_l(\underline{r}) = r^l \sum C_{\alpha\beta\gamma} \left(\frac{x}{r}\right)^\alpha \left(\frac{y}{r}\right)^\beta \left(\frac{z}{r}\right)^\gamma$$

<u>Def</u>: $\mathcal{P}_l \equiv$ vector space of all complex homogeneous polys of degree l in R_3.
$\mathcal{P}_l(\Omega) \equiv$ vector space of all functions $P_l : \Omega \to C$ defined by restricting $P_l \in \mathcal{P}_l$ to Ω. Every element $P_l(\hat{r}) \in \mathcal{P}_l(\Omega)$ gives rise to a unique $r^l P_l(\hat{r}) = P_l(\underline{r}) \in \mathcal{P}_l$.
$\mathcal{P}_l(\Omega)$ is a <u>subspace</u> of $\mathcal{L}_2(\Omega)$, since it is a vector space and since every $P_l(\hat{r})$ is certainly square integrable.

What is the <u>dimension</u> of $\mathcal{P}_\ell$ and $\mathcal{P}_\ell(\Omega)$?
Consider $\alpha + \beta + \gamma = \ell$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$

$\alpha = 0$, $\ell+1$ choices for β, γ determined
$\alpha = 1$, ℓ " " , " "
$\vdots$
$\alpha = \ell$, 1 " " , " "

(The <u>question</u> is: given α, β, γ integers ≥ 0 s.t. $\alpha + \beta + \gamma = \ell$, how many such combinations are there)

$$\dim \mathcal{P}_\ell = \dim \mathcal{P}_\ell(\Omega) = 1 + 2 + \cdots + (\ell+1) = \frac{(\ell+1)(\ell+2)}{2}.$$

<u>Def:</u> a scalar function $\phi(\underline{r})$ $\phi : \mathbb{R}^3 \to \mathbb{C}$ is said to be <u>harmonic</u> in an open set V of $\mathbb{R}^3$ if $\nabla^2 \phi = 0$ in V.
Example: if $\phi(\underline{r})$ is the grav. pot. of some mass distr. lying outside a (closed) volume V of $\mathbb{R}^3$, then ϕ is harmonic in $E - V$.

V $\quad \nabla^2 \phi = 0$ outside of V.

<u>Def:</u> $\mathcal{H}_\ell \equiv$ the <u>vector space</u> of all harmonic members of $\mathcal{P}_\ell$. Note: if a poly. is harmonic in any open set, it is harmonic everywhere.

Note: in general if $P_\ell \in \mathcal{P}_\ell$, then

$\nabla^2 P_\ell \in \mathcal{P}_{\ell-2}$, since $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$

Def : $\mathcal{H}_\ell(\Omega) \equiv$ space of all $Y_\ell : \Omega \to \mathbb{C}$ defined by restricting the elements $H_\ell(\underline{r}) \in \mathcal{H}_\ell$ to Ω.

Then $\mathcal{H}_\ell(\Omega)$ is a subspace of $\mathcal{L}_2(\Omega)$. It is in fact the space we wish to study because

in D&T we call this $Y_\ell(r)$

> Def: Any member $H_\ell(\underline{r})$ of $\mathcal{H}_\ell$ (i.e. any harmonic, homogeneous polynomial of degree ℓ is called a solid spherical harmonic of degree ℓ. The element $Y_\ell(\hat{r}) \in \mathcal{H}_\ell(\Omega)$ defined by $H_\ell(\underline{r}) = r^\ell Y_\ell(\hat{r})$ is called the surface spherical harmonic of degree ℓ determined by H_ℓ.

Our main task will be to construct an orthonormal basis for the spaces $\mathcal{H}_\ell$ and $\mathcal{H}_\ell(\Omega)$. First give examples : $\ell = 0$ and $\ell = 1$ also $\ell = 2$

19. The operators ∇, ∇_1 and Λ

The key to examining the structure of $\mathcal{H}_\ell$ and $\mathcal{H}_\ell(\Omega)$ will be the operator $\underline{\Lambda}$, already discussed this term by N. Frazer. We consider first a more familiar operator.

<u>Def</u>: Let $f(\underline{r})$ be a scalar field defined in an open set V of R_3 $\quad f: V \to C$. The field f is said to be <u>differentiable</u> at $\underline{r} \in V$ if $\exists$ a vector $\underline{F}(\underline{r})$ depending on $\underline{r}$ s.t.

$$f(\underline{r}+\underline{\delta r}) = f(\underline{r}) + \underline{\delta r}\cdot \underline{F}(\underline{r}) + O(|\underline{\delta r}|^2)$$

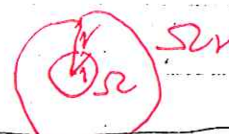
$E(\underline{r},\underline{\delta r})$ s.t.

$E/|\underline{\delta r}| \to 0$ as $|\underline{\delta r}| \to 0$.

Section A.7.1

r_1 and $\Lambda = \hat{r} \times r_1$

This is the usual (coordinate-free) defn of the <u>gradient</u> of a scalar field. $\underline{F}(\underline{r})$ is usually written $\underline{\nabla} f(\underline{r})$ and is called the <u>gradient</u> of f at $\underline{r}$.



$$\boxed{\text{In spherical coordinates} \quad \underline{\nabla} = \hat{r}\,\partial_r + r^{-1}\left(\hat{\theta}\,\partial_\theta + \hat{\phi}\,\frac{1}{\sin\theta}\,\partial_\phi\right)}$$

Let Ω_r be the surface of the sphere of radius r centered on $\underline{0}$ in R_3. Then $\Omega_1 \equiv \Omega$, the surface of the unit sphere.

Consider the vector $\underline{\nabla} f - \hat{r}\,(\hat{r}\cdot\underline{\nabla} f)$

At every point $\underline{r} = r\hat{r}$ on Ω_r, this vector is tangent to the sphere Ω_r.

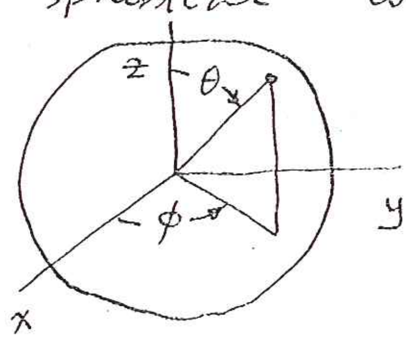
Let $\quad \underline{\nabla}_r f(\underline{r}) \equiv \underline{\nabla} f(\underline{r}) - \hat{r}[\hat{r} \cdot \underline{\nabla} f(\underline{r})]$

Then ∇_r is called the surface gradient operator on Ω_r.

Can compute $\nabla_r f$ knowing only values of f on Ω_r.

Geometrically $\underline{\nabla} f(\underline{r})$ is rate of change of f at $\underline{r}$ in direction of maximum change. Likewise $\nabla_r f(\underline{r})$ is rate of change of f along the surface Ω_r in the direction of maximum change along that surface

∇_1 is the surface gradient operator on Ω. In spherical coordinates



$$\nabla = \hat{r}\partial_r + \frac{1}{r}\left[\hat{\theta}\partial_\theta + \frac{1}{\sin\theta}\hat{\phi}\partial_\phi\right]$$

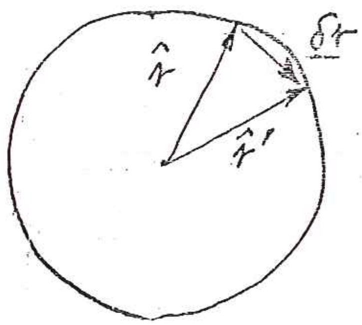
$$\boxed{\begin{aligned} \nabla_r &= r^{-1}\nabla_1 \\ \nabla_1 &= \hat{\theta}\partial_\theta + (\sin\theta)^{-1}\hat{\phi}\partial_\phi \end{aligned}}$$

coordinate free: $\nabla = \hat{r}\partial_r + r^{-1}\nabla_1$

The gradient ∇ may be considered a <u>linear operator</u> which maps a differentiable scalar field f in a region V of $\mathbb{R}_3$ into a vector field $\underline{\nabla f} \in V$.

The surface gradient ∇_1 is a linear operator which maps a scalar field $f : \Omega \to \mathbb{C}$ which is differentiable on Ω into a <u>tangent</u> <u>vector</u> field $\nabla_1 f$ on Ω.

Note: consider two unit vectors $\hat{r}'$ and $\hat{r}$ as $|\hat{r}' - \hat{r}| \to 0$. Their difference $\hat{r}' - \hat{r} \to$ a tangent vector.



$$\hat{r}' = \hat{r} + \underline{\delta r}$$

$$f(\hat{r} + \underline{\delta r}) = f(\hat{r}) + \underline{\delta r} \cdot \nabla f(\hat{r}) + O(|\underline{\delta r}|^2)$$
$$= f(\hat{r}) + \underline{\delta r} \cdot \nabla_1 f(\hat{r}) + O(|\underline{\delta r}|^2)$$

$$\boxed{f(\hat{r} + \underline{\delta r}) = f(\hat{r}) + \underline{\delta r} \cdot \nabla_1 f(\hat{r}) + O(|\underline{\delta r}|^2) \quad \text{as } |\underline{\delta r}| \to 0 \qquad \star}$$

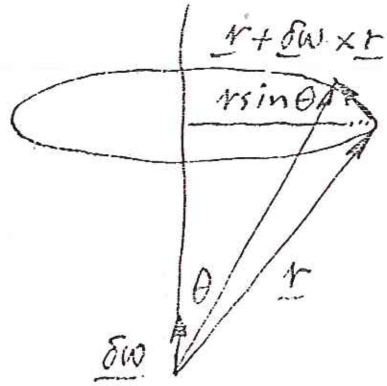
- The surface gradient ∇_1 enables us to calculate f at nearby points $\hat{r} + \underline{\delta r}$ on Ω via $\star$ above.

A related operator which also maps a differentiable scalar field f into a vector field is $\underline{\Lambda}$, maps f into $\underline{\Lambda} f$. Consider

$$f(\underline{r} + \underline{\delta\omega} \times \underline{r}) = f(\underline{r}) + \underline{\delta\omega} \times \underline{r} \cdot \underline{\nabla} f(\underline{r}) + O(|\underline{\delta\omega}|^2)$$
$$= f(\underline{r}) + \underline{\delta\omega} \cdot \underline{r} \times \nabla f(\underline{r}) + O(|\underline{\delta\omega}|^2)$$

Define $\underline{\Lambda} \equiv \underline{r} \times \underline{\nabla}$. Then
$$f(\underline{r} + \underline{\delta\omega} \times \underline{r}) = f(\underline{r}) + \underline{\delta\omega} \cdot \underline{\Lambda} f(\underline{r}) + O(|\underline{\delta\omega}|^2)$$
☆☆

$\underline{\Lambda}$ can be called the <u>infinitesimal rotation operator</u>



$\underline{\delta\omega} \times \underline{r}$ is $\perp$ to both $\underline{\delta\omega}$ and $\underline{r}$

$|\underline{\delta\omega} \times \underline{r}| = |\underline{\delta\omega}||\underline{r}| \sin\theta$

Every $\underline{r} + \underline{\delta\omega} \times \underline{r}$ is obtained from $\underline{r}$ by a rigid rotation of all of space through the small angle $|\underline{\delta\omega}|$ radians about the axis $\underline{\delta\omega}$.

The infinitesimal rotation operator $\underline{r} \times \underline{\nabla} \equiv \underline{\Lambda}$ enables us to calculate f at infinitesimally rotated points $\underline{r} + \delta\underline{\omega} \times \underline{r}$ via ** above. Hence its name. Similarly one could call $\underline{\nabla}$ the inf. translation operator as it enable us to compute f at infinitesimally translated points $\underline{r} + \underline{\delta r}$.

Note that like $\underline{\nabla}_r f(\underline{r})$, $\underline{\Lambda} f(\underline{r})$ is a tangent vector to Ω_r, and $\underline{\Lambda} f(\underline{r})$ can be calculated at every $\underline{r} \in \Omega_r$ knowing only values of f on Ω_r.

$$\underline{\Lambda} = \underline{r} \times \underline{\nabla} = \underline{r} \times \left[\hat{r}\, \partial_r + \frac{1}{r}\, \hat{\theta}\, \partial_\theta + \frac{1}{r\sin\theta}\, \hat{\phi}\, \partial_\phi \right].$$

$$= -\hat{\theta}\, \frac{1}{\sin\theta}\, \partial_\phi + \hat{\phi}\, \partial_\theta$$

$$\underline{\Lambda} = \underline{r} \times \underline{\nabla} = \underline{r} \times \left[\hat{r}\, \partial_r + r^{-1}\, \underline{\nabla}_1 \right] = \hat{r} \times \underline{\nabla}_1$$

$$\boxed{\begin{aligned} \underline{\Lambda} &= \hat{r} \times \underline{\nabla}_1 \\ &= \hat{\theta}\left(-\frac{1}{\sin\theta}\, \partial_\phi\right) + \hat{\phi}\, \partial_\theta \end{aligned}}$$

$\underline{\nabla}$ acts in R_3 (or some open region V thereof).

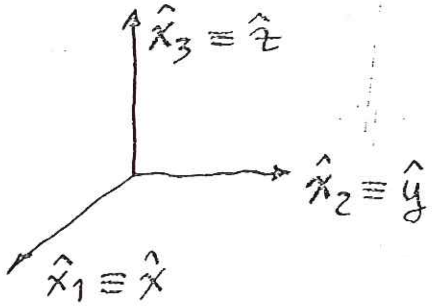
∇_1 acts on Ω.

$\underline{\Lambda}$ can be thought of as acting either in R_3 or on Ω (or on any Ω_r).

$\underline{\nabla}_r$ acts on Ω_r (and is given in terms of $\underline{\nabla}_1$ on $\Omega_1 \equiv \Omega$ by $\nabla_r = \frac{1}{r}\, \nabla_1$).

20. Properties of the Λ operator

First, those easy in Cartesian coordinates.



Define three scalar operators

$$\boxed{\underline{\Lambda} = \Lambda_i \hat{x}_i \quad , \quad \Lambda_i = \hat{x}_i \cdot \underline{\Lambda}}$$

$\underline{\Lambda} = \underline{r} \times \underline{\nabla}$, so

$\Lambda_i = \varepsilon_{ijk} r_j \partial_k$

Let us sometimes use:
$\hat{x} \equiv \hat{x}_1$, $\hat{y} \equiv \hat{x}_2$, $\hat{z} \equiv \hat{x}_3$. Then

$\underline{\Lambda} = \hat{x} \Lambda_x + \hat{y} \Lambda_y + \hat{z} \Lambda_z$ where

$$\Lambda_x = y \partial_z - z \partial_y$$
$$\Lambda_y = z \partial_x - x \partial_z$$
$$\Lambda_z = x \partial_y - y \partial_x$$

Now examine $\Lambda^2 = \underline{\Lambda} \cdot \underline{\Lambda} = \Lambda_i \Lambda_i$
(Λ^2 is to $\underline{\Lambda}$ as $\nabla^2 = \underline{\nabla} \cdot \underline{\nabla}$ is to $\underline{\nabla}$).

$$\begin{aligned}
\Lambda_i \Lambda_i &= (\varepsilon_{ijk} r_j \partial_k)(\varepsilon_{ilm} r_l \partial_m) \\
&= \varepsilon_{ijk} \varepsilon_{ilm} r_j \partial_k (r_l \partial_m) \\
&= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) r_j \partial_k (r_l \partial_m) \\
&= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl})(r_j r_l \partial_k \partial_m + \delta_{kl} r_j \partial_m)
\end{aligned}$$

$$= r^2 \nabla^2 - r_j r_k \partial_j \partial_k - 2 r_j \partial_j$$

$$\boxed{\Lambda^2 = r^2 \nabla^2 - r_k \partial_k (r_j \partial_j + 1)}$$

Now examine $\underline{\Lambda} \times \underline{\Lambda}$

$$(\underline{\Lambda} \times \underline{\Lambda})_i = \varepsilon_{ijk} (\varepsilon_{jlm} r_l \partial_m) (\varepsilon_{kp\underline{q}} r_p \partial_{\underline{q}})$$

$$= \varepsilon_{ijk} \varepsilon_{jlm} \varepsilon_{kp\underline{q}} \; r_l \partial_m (r_p \partial_{\underline{q}})$$

$$= (\delta_{ip} \delta_{j\underline{q}} - \delta_{i\underline{q}} \delta_{jp}) \varepsilon_{jlm} (r_l r_p \partial_m \partial_{\underline{q}} + r_l \delta_{mp} \partial_{\underline{q}})$$

$$= (\delta_{ip} \varepsilon_{\underline{q}lm} - \delta_{i\underline{q}} \varepsilon_{plm}) (r_l r_p \partial_m \partial_{\underline{q}} + r_l \delta_{mp} \partial_{\underline{q}})$$

since symmetric . antisymmetric = 0

$$= \delta_{ip} \varepsilon_{\underline{q}lm} r_l \delta_{mp} \partial_{\underline{q}} = \varepsilon_{\underline{q}li} r_l \partial_{\underline{q}}$$

$$= - \varepsilon_{il\underline{q}} r_l \partial_{\underline{q}} = - \Lambda_i$$

$$\boxed{\underline{\Lambda} \times \underline{\Lambda} = - \underline{\Lambda}}$$

i.e. $\quad \Lambda_x \Lambda_y - \Lambda_y \Lambda_x = - \Lambda_z$, etc.

We can write this in a familiar notation:

Define the <u>commutator</u> of any two linear operators A and B (both acting on the same vector space V) to be,

$$[A, B] \equiv AB - BA$$

Note $[A, B] = -[B, A]$

The above is equivalent to

$$\boxed{\begin{aligned} [\Lambda_x, \Lambda_y] &= -\Lambda_z \\ [\Lambda_y, \Lambda_z] &= -\Lambda_x \\ [\Lambda_z, \Lambda_x] &= -\Lambda_y \\ [\Lambda_i, \Lambda_j] &= -\varepsilon_{ijk}\Lambda_k \end{aligned}}$$

Now define $\boxed{\begin{aligned} \Lambda_+ &\equiv \Lambda_x + i\Lambda_y \\ \Lambda_- &\equiv \Lambda_x - i\Lambda_y \end{aligned}}$

Commutation relations for $\Lambda_\pm$ are

$$\boxed{\begin{aligned} [\Lambda_+, \Lambda_-] &= 2i\Lambda_z \\ [\Lambda_+, \Lambda_z] &= -i\Lambda_+ \\ [\Lambda_-, \Lambda_z] &= i\Lambda_- \end{aligned}}$$

Look at $(\Lambda_x + i\Lambda_y)(\Lambda_x - i\Lambda_y)$

$$= \Lambda_x^2 + \Lambda_y^2 + i(\Lambda_y\Lambda_x - \Lambda_x\Lambda_y)$$
$$= \Lambda_x^2 + \Lambda_y^2 + i\Lambda_z$$

Thus:

$$\boxed{\begin{array}{l} \Lambda^2 = \Lambda_+\Lambda_- + \Lambda_z^2 - i\Lambda_z \\ \text{likewise}\quad \Lambda^2 = \Lambda_-\Lambda_+ + \Lambda_z^2 + i\Lambda_z \end{array}}$$

Consider now ∇^2 and $\underline{\Lambda}$. Claim $[\nabla^2, \underline{\Lambda}] = 0$.

Proof:

$$\partial_\ell\partial_\ell(\varepsilon_{ijk}r_j\partial_k) = \varepsilon_{ijk}\partial_\ell(\delta_{j\ell}\partial_k + r_j\partial_\ell\partial_k)$$

$$= \varepsilon_{ijk}(\delta_{j\ell}\partial_\ell\partial_k + \delta_{j\ell}\partial_\ell\partial_k + r_j\partial_\ell\partial_\ell\partial_k)$$

$$= \cancel{2\varepsilon_{i\ell k}\partial_\ell\partial_k} \to 0 + \varepsilon_{ijk}r_j\partial_k\partial_\ell\partial_\ell$$

$$= (\varepsilon_{ijk}r_j\partial_k)\partial_\ell\partial_\ell$$

Thus ∇^2 and $\underline{\Lambda}$ commute, i.e.

$$\boxed{[\nabla^2, \underline{\Lambda}] = 0}$$

Now we consider the properties most easily obtained in <u>spherical</u> coordinates

We have seen that in $\hat{r}, \hat{\theta}, \hat{\phi}$

$$\boxed{\underline{\Lambda} = -\hat{\theta}(\sin\theta)^{-1}\partial_\phi + \hat{\phi}\,\partial_\theta}$$

Thus $\underline{\Lambda}$ commutes with ∂_r and any function of r alone.

$$\boxed{\begin{aligned}[\partial_r, \underline{\Lambda}] &= 0 \\ [f(r), \underline{\Lambda}] &= 0\end{aligned}}$$

Also $\underline{\Lambda} f(\underline{r}) = \underline{\Lambda} f(r,\theta,\phi)$ depends only the values of f on Ω_r, as noted above.

We now make use of

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$
$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\underline{\Lambda} = \hat{\theta}\left(-\frac{1}{\sin\theta}\partial_\phi\right) + \hat{\phi}\,\partial_\theta = \hat{x}\Lambda_x + \hat{y}\Lambda_y + \hat{z}\Lambda_z$$

$$\begin{aligned} &= \hat{x}\,(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi) \\ &\quad + \hat{y}\,(\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi) \\ &\quad + \hat{z}\,\partial_\phi \end{aligned}$$

Thus $\boxed{\Lambda_z = \partial_\phi}$

$$\Lambda_x = -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\phi$$
$$\Lambda_y = \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi$$

Thus

$$\boxed{\begin{aligned} \Lambda_+ &= ie^{i\phi}\left(\partial_\theta + i\cot\theta\, \partial_\phi\right) \\ \Lambda_- &= ie^{-i\phi}\left(-\partial_\theta + i\cot\theta\, \partial_\phi\right) \end{aligned}}$$

Now from above $\Lambda^2 = \Lambda_+\Lambda_- + \Lambda_z^2 - i\Lambda_z$

$$= e^{i\phi}\left(\partial_\theta + i\cot\theta\, \partial_\phi\right) e^{-i\phi}\left(\partial_\theta - i\cot\theta\, \partial_\phi\right) + \partial_\phi^2 - i\partial_\phi$$

$$= \partial_\theta^2 + i\left(\csc^2\theta - \cot^2\theta\right)\partial_\phi + \cot\theta\, \partial_\theta \qquad \left(\frac{\partial}{\partial\theta}(\cot\theta)\right.$$
$$+ \left(\cot^2\theta + 1\right)\partial_\phi^2 - i\partial_\phi \qquad \left. = -\csc^2\theta\right)$$

$$\csc^2\theta = 1 + \cot^2\theta$$

$$= \partial_\theta^2 + \cot\theta\, \partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2$$

$$\boxed{\begin{aligned} \Lambda^2 &= \partial_\theta^2 + \cot\theta\, \partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 \\ &= \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\, \partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\phi^2 \\ &\qquad\qquad \star \end{aligned}}$$

Now claim that $[\underline{\Lambda}, \Lambda^2] = 0$

Proof: suffices to show for Λ_z since choice of

$\hat{x}, \hat{y}, \hat{z}$ is arbitrary.

But $\Lambda_z = \partial_\phi$, and this clearly commutes with $\star$. Thus

$$\boxed{[\underline{\Lambda}, \Lambda^2] = 0}$$

Check it directly for Λ_x, Λ_y if you don't believe it.

The combination $\star$ of ∂_θ, ∂_ϕ might look familiar. It is because

$$\boxed{\nabla^2 = \frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\Lambda^2}$$

Proof: we have shown that

$$\begin{aligned}\Lambda^2 &= r^2\nabla^2 - r_k\partial_k(r_j\partial_j + 1) \\ &= r^2\nabla^2 - r\partial_r(r\partial_r + 1)\end{aligned}$$

$$\left(\text{since } r_i\partial_i = \underline{r}\cdot\underline{\nabla} = \underline{r}\cdot\left(\hat{r}\partial_r + \frac{1}{r}\hat{\theta}\partial_\theta + \frac{1}{r\sin\theta}\hat{\phi}\partial_\phi\right) = r\partial_r\right).$$

$$= r^2\nabla^2 - r\partial_r^2 r \quad , \qquad \text{Thus}$$

$$\nabla^2 = \frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\Lambda^2.$$

Can now offer a more direct proof of $[\nabla^2, \underline{\Lambda}] = 0$, viz.

$$\nabla^2 = \frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\Lambda^2 \quad \text{and}$$
$$[\underline{\Lambda}, \Lambda^2] = 0$$
$$[\underline{\Lambda}, \partial_r] = 0$$
$$[\underline{\Lambda}, f(r)] = 0$$

, i.e. $\underline{\Lambda}$ commutes with Λ^2, ∂_r and any $f(r)$.

Thus $\underline{\Lambda}$ commutes with ∇^2, a comb. of Λ^2, ∂_r and $f(r)$.

call this Y_ℓ

Finally, let us apply Λ^2 to $H_\ell(\underline{r}) \in \mathcal{H}_\ell$, a <u>solid</u> <u>spherical</u> <u>harmonic</u>.

$$\nabla^2 \overset{Y_\ell}{H_\ell} = 0, \quad \text{so}$$

$$\frac{1}{r}\partial_r^2 r\,[r^\ell \overset{Y_\ell}{H_\ell}(\hat{r})] + \frac{1}{r^2}\Lambda^2[r^\ell \overset{Y_\ell}{H_\ell}(\hat{r})] = 0$$

$$(\ell+1)\ell\, r^{\ell-2}\overset{Y_\ell}{H_\ell}(\hat{r}) + r^{\ell-2}\Lambda^2 \overset{Y_\ell}{H_\ell}(\hat{r}) = 0$$

divide by $r^{\ell-2}$.

call this Y_ℓ

$$\boxed{\begin{aligned} \Lambda^2 H_\ell(\hat{r}) &= -\ell(\ell+1) H_\ell(\hat{r}) \\ \Lambda^2 H_\ell(\underline{r}) &= -\ell(\ell+1) H_\ell(\underline{r}) \end{aligned}}$$

We used both harmonicness and homogeneousness of degree ℓ.

21. The angular momentum operator L

Defn: If V is a finite-dimensional inner product space and $B: V \to V$, then the adjoint of B, written B^+, is that unique linear operator s.t. $\forall f, g \in V$,

both can be easily shown

$$(f, Bg) = (B^+ f, g)$$ use $\langle \psi, BX \rangle$

If the matrix of B w.r.t. a basis $\{v_i\}$ is B, then the matrix of B^+ is B^+, defined by $(B^+)_{ij} = B_{ji}{}^*$.

Other properties:

$(B^+)^+ = B$ proof: $(f, B^+ g) = (B^+ g, f)^*$
$= (g, Bf)^* = (Bf, g)$
also $\equiv (B^{++} f, g)$

$(aB)^+ = a^* B^+$
$(AB)^+ = B^+ A^+$

Defn: An operator $B: V \to V$ is self-adjoint or Hermitian if $B^+ = B$.
If $B^+ = -B$, B is anti-Hermitian.

Every linear operator is the sum of a Hermitian and an anti-Hermitian operator.

Question: is $\underline{\Lambda}$ Hermitian

Answer: no, but it is <u>anti</u>-Hermitian.

This leads us to <u>define</u>

$$\boxed{\underline{L} \equiv -i\underline{\Lambda} = \frac{1}{i}\underline{\Lambda} = \frac{1}{i}\underline{r} \times \underline{\nabla}}$$

Note: $\hbar \underline{L}$ is the quantum-mechanical angular momentum operator. The three components of $\underline{L}$ are <u>observables</u>.

We <u>now</u> wish to show that $\underline{L}$ maps $\mathcal{P}_\ell$ into itself, $\mathcal{H}_\ell$ into itself and that (in both cases) $\underline{L}$ is <u>Hermitian</u>, i.e.

$$\boxed{\begin{array}{l} \underline{L} : \mathcal{P}_\ell \to \mathcal{P}_\ell \\ \underline{L} : \mathcal{H}_\ell \to \mathcal{H}_\ell \\ \underline{L}^{+} = \underline{L} \end{array}}$$

This a main <u>theorem</u>.

$\underline{L} : \mathcal{Y}_\ell \to \mathcal{Y}_\ell$

Proof: suffices to prove for L_z

1. $L_z : \mathcal{P}_\ell \to \mathcal{P}_\ell$ since $L_z = \frac{1}{i}(x\partial_y - y\partial_x)$.
   Thus if $P_\ell \in \mathcal{P}_\ell$ so is $L_z P_\ell$.
   Same clearly true for L_x, L_y.

2. $L_z : \mathcal{H}_\ell \to \mathcal{H}_\ell$. If $H_\ell \in \mathcal{H}_\ell$ then $L_z H_\ell \in \mathcal{P}_\ell$. Does $\nabla^2 L_z H_\ell = 0$?
Yes, since $\nabla^2 L_z H_\ell = \frac{1}{i} \nabla^2 \Lambda_z H_\ell$
$= \frac{1}{i} \Lambda_z \nabla^2 H_\ell = \frac{1}{i} \Lambda_z 0 = 0$.
Same clearly true for L_x, L_y.

3. We must show $\forall f, g \in \mathcal{P}_\ell$ that $(f, L_z g) = (L_z f, g)$. But

$$(f, L_z g) = \int_\Omega f \, (L_z g)^* \, dA$$ use $\langle \psi, L_z \chi \rangle$

$$= \int_\Omega f \left(\frac{1}{i} \partial_\phi g \right)^* dA$$

$$= -\frac{1}{i} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \; f \, \partial_\phi g^*$$

integrate by parts

$$= \frac{1}{i} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \; (\partial_\phi f) \, g^*$$

$$= \frac{1}{i} \int_\Omega (\partial_\phi f) \, g^* \, dA$$

$$= \int_\Omega \left(\frac{1}{i} \partial_\phi f \right) g^* \, dA$$

$$= \int_\Omega (L_z f) \, g^* \, dA = (L_z f, g)$$ q.e.d.

Clearly $\underline{L} : \mathcal{P}_\ell \to \mathcal{P}_\ell$ Hermitian $\Rightarrow$
$\underline{L} : \mathcal{H}_\ell \to \mathcal{H}_\ell$ Hermitian

<u>Corollaries:</u>

$\boxed{\text{$L^2 = -\Lambda^2$ is also self-adjoint on both $\mathcal{P}_\ell$ and $\mathcal{H}_\ell$.}}$

proof: $L^2 = L_i L_i$ and $(AB)^+ = B^+ A^+$
so $(L^2)^+ = (L_i L_i)^+ = L_i^+ L_i^+ = L_i L_i$
$= L^2$

$L_+ = \frac{1}{i} \Lambda_+$ and $L_- = \frac{1}{i} \Lambda_-$ are Hermitian conjugates on both $\mathcal{P}_\ell$ and $\mathcal{H}_\ell$, i.e.

$$(L_+)^+ = L_-$$
$$(L_-)^+ = L_+$$

proof: $(L_+)^+ = (L_x + iL_y)^+ = L_x^+ - iL_y^+$
$= L_x - iL_y = L_-$

Useful $\underline{L}$ identities and commutators are:

$$\underline{L} \times \underline{L} = i\underline{L}$$
$$L^2 = L_+ L_- + L_z^2 - L_z = L_- L_+ + L_z^2 + L_z$$
$$[L^2, \underline{L}] = [L^2, L_z] = [L^2, L_\pm] = 0$$
$$[L_z, L_+] = L_+ \quad , \quad [L_z, L_-] = -L_- \quad , \quad [L_+, L_-] = 2L_z$$

Recall the quantum mechanical significance of commutativity: two observables which commute may be simultaneously specified.

Thus $[L^2, L_z] = 0$ means that L^2 and L_z can be simultaneously specified, i.e. the squared length of the angular momentum and any component. Two orthogonal components may however not be simultaneously specified exactly.

end lecture 14 here

22. Algebraic structure of the space $\mathcal{P}_\ell$.

Defn: Two subspaces $\mathcal{U}$ and $\mathcal{V}$ of an inner product space W are orthogonal if $\forall u \in \mathcal{U}$ and $\forall v \in \mathcal{V}$, $(u, v) = 0$.

We write $\mathcal{U} \perp \mathcal{V}$.

$$\boxed{\text{If } \ell \neq \ell', \text{ then } \mathcal{H}_\ell \perp \mathcal{H}_{\ell'}}$$

i.e. spherical harmonics of different degrees ℓ are orthogonal.

proof: let $H_\ell \in \mathcal{H}_\ell$ and $H_{\ell'} \in \mathcal{H}_{\ell'}$.
Consider $(H_\ell, \Lambda^2 H_{\ell'}) = (\Lambda^2 H_\ell, H_{\ell'})$
since Λ^2 is Hermitian. call this $Y_\ell, Y_{\ell'}$

$$-\ell'(\ell'+1)(H_\ell, H_{\ell'}) = -\ell(\ell+1)(H_\ell, H_{\ell'})$$

$$[\ell(\ell+1) - \ell'(\ell'+1)](H_\ell, H_{\ell'}) = 0 \quad \text{or}$$

$$(\ell'-\ell)\underbrace{(\ell'+\ell+1)}_{\geq 0}(H_\ell, H_{\ell'}) = 0$$

$Y_\ell, Y_{\ell'}$

Thus $(H_\ell, H_{\ell'}) = 0$ if $\ell' \neq \ell$.

Now define $r^\nu \mathcal{H}_\ell$ to be the space of all fns of the form $r^\nu H_\ell$ where $H_\ell \in \mathcal{H}_\ell$.

The main theorem for the space $\mathcal{P}_\ell$ is: that it has a direct sum decomposition of the form

$\oplus \begin{cases} r^\ell Y_0 & \ell \text{ even} \\ r^{\ell-1} Y_1 & \ell \text{ odd} \end{cases}$

$$\mathcal{P}_\ell = \overset{\mathcal{H}_\ell \;\; Y_\ell}{\mathcal{H}_\ell} \oplus r^2 \overset{Y_{\ell-2}}{\mathcal{H}_{\ell-2}} \oplus \cdots \oplus r^{\ell-\tau(\ell)} \overset{Y_{\tau(\ell)}}{\mathcal{H}_{\tau(\ell)}}$$

where $\tau(\ell) = \ell \bmod 2 = 0$ if ℓ even, 1 if odd

The assertion is that for any $P_\ell \in \mathcal{P}_\ell$, $\exists$ unique solid spherical harmonics

H_ℓ , $H_{\ell-2}$, ... s.t.

$$\boxed{\begin{gathered} P_\ell = H_\ell + r^2 H_{\ell-2} + \cdots + r^{\ell-\tau(\ell)} H_{\tau(\ell)} \\ \tau(\ell) = \left\{ \begin{matrix} 0 & \ell \text{ even} \\ 1 & \ell \text{ odd} \end{matrix} \right\} = \ell \bmod 2 \qquad * \end{gathered}}$$

This theorem lies at the ♡ of making multipole expansions of non-harmonic fcns.

Study of P_ℓ's is thereby reduced to study of H_ℓ's. Recall that the H_ℓ's in * are mutually orthogonal.

Proof:

(i) uniqueness : say $\tilde{P}_\ell$ and $\tilde{\tilde{P}}_\ell$ both have expansions of the form *. Then so does their difference $P_\ell \equiv \tilde{P}_\ell - \tilde{\tilde{P}}_\ell$. Look on Ω. If $P_\ell = 0$ on Ω then $(P_\ell, H_{\ell-2\nu}) = (0, H_{\ell-2\nu}) = 0$ but because $H_{\ell'} \perp H_\ell$, $(P_\ell, H_{\ell-2\nu})$ $= r^{2\nu}(H_{\ell-2\nu}, H_{\ell-2\nu}) = 0$ for any ν. Hence $H_{\ell-2\nu} = 0$ for any ν. Hence $\tilde{H}_{\ell-2\nu} = \tilde{\tilde{H}}_{\ell-2\nu}$ for any ν.

(ii) to prove existence, we need a lemma

$$\boxed{\nabla^2 r^{n+2} H_\ell = (n+2)(2\ell+n+3)\, r^n H_\ell}$$

proof: $\nabla^2 r^{n+2} H_\ell(\underline{r}) = [\frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\Lambda^2]\, r^{\ell+n+2} H_\ell(\hat{r})$

$$= \frac{1}{r}\partial_r^2 r^{\ell+n+3} H_\ell(\hat{r}) + r^{\ell+n}\Lambda^2 H_\ell(\hat{r})$$

$$= [(\ell+n+3)(\ell+n+2) - \ell(\ell+1)]\, r^{\ell+n} H_\ell(\hat{r})$$

$$= (n+2)(2\ell+n+3)\, r^{\ell+n} H_\ell(\hat{r})$$

$$= (n+2)(2\ell+n+3)\, r^n H_\ell(\underline{r})$$

(iii) now prove existence by <u>induction</u>

true for $\ell = 0$: $\mathcal{P}_0 = H_0$, 1 a basis
true for $\ell = 1$: $\mathcal{P}_1 = H_1$, x, y, z a basis

i.e. every homog. poly of degree 0 or 1 is harmonic.

Now suppose true for ℓ. Want to prove for $\ell+2$. Consider $P_{\ell+2} \in \mathcal{P}_{\ell+2}$.

We know that $\nabla^2 P_{\ell+2} \in \mathcal{P}_\ell$ so, by assumption, $\exists$ spherical harmonics $H_\ell, \ldots H_{\pi(\ell)}$ s.t.

$$\nabla^2 P_{\ell+2} = H_\ell + r^2 H_{\ell-2} + \cdots \quad r^{\ell-\tau(\ell)} H_{\tau(\ell)}$$

but by the lemma, the r.h.s. is

$$= \nabla^2 \left[\frac{r^2 H_\ell}{2(2\ell+3)} + \frac{r^4 H_{\ell-2}}{4(2\ell+1)} + \frac{r^6 H_{\ell-4}}{6(2\ell-1)} + \cdots \right.$$

$$\left. \cdots + \frac{r^{\ell+2-\tau(\ell)} H_{\tau(\ell)}}{(\ell+2-\tau(\ell))(\ell+3-\tau(\ell))} \right]$$

Let $H_\ell' \equiv \dfrac{H_\ell}{2(2\ell+3)}$, $H_{\ell-2}' = \dfrac{H_{\ell-2}}{4(2\ell+1)}$, etc.

Then

$$\nabla^2 [\underbrace{P_{\ell+2} - r^2 H_\ell' - r^4 H_{\ell-2}' - \cdots \quad r^{\ell+2-\tau(\ell)} H_{\tau(\ell)}'}_{F_{\ell+2}}] = 0$$

Thus we can write $P_{\ell+2}$ in the form

$$P_{\ell+2} = F_{\ell+2} + r^2 H_\ell' + \cdots \quad r^{\ell+2-\tau(\ell)} H_{\tau(\ell)}'$$

where $F_{\ell+2} \in \mathcal{P}_{\ell+2}$ (since $P_{\ell+2}$ and all of $r^2 H_\ell'$, etc. are) and where $\nabla^2 F_{\ell+2} = 0$.

Hence $F_{\ell+2}$ is harmonic, i.e. $F_{\ell+2} = H_{\ell+2}' \in \mathcal{H}_{\ell+2}$.

Also $\tau(\ell+2) = \tau(\ell)$. Hence

$$P_{\ell+2} = H'_{\ell+2} + r^2 H'_\ell + \cdots \quad r^{\ell+2-\tau(\ell+2)} H'_{\tau(\ell+2)}$$

and we are done.

Any poly $F(\underline{r})$ can be written as a sum of homogeneous polys, so we have the immediate <u>corollary</u>.

<u>Any</u> poly $F(\underline{r})$ can be written as a finite sum of solid spherical harmonics

$$F(\underline{r}) = \sum_{\ell=0}^{L} \sum_{n=0}^{N} r^n H_\ell^{(n)}(\underline{r}) \quad \text{where}$$

$$H_\ell^{(n)} \in \mathcal{H}_\ell .$$

Similarly if $F(\hat{r})$ is any polynomial mapping $\Omega \to \mathbb{C}$, then $F(\hat{r})$ can be written as a finite sum of surface spherical harmonics $Y_\ell(\hat{r}) \in \mathcal{H}_\ell(\Omega)$

$$\boxed{F(\hat{r}) = \sum_{\ell=0}^{L} Y_\ell(\hat{r}) \quad \text{where} \quad Y_\ell \in \mathcal{H}_\ell(\Omega)}$$

This expansion is <u>clearly</u> <u>unique</u>. The same uniqueness proof works.

So any poly $F: \Omega \to \mathbb{C}$ can be written as a unique finite sum of surface spherical harmonics Y_ℓ on Ω.

It is now a simple matter to prove the following <u>expansion theorem</u>.

<u>Theorem</u> : if $f: \Omega \to \mathbb{C}$ is a <u>continuous</u> fcn, then $\forall\ \epsilon > 0\ \exists\ L$ and a sequence of spherical harmonics $Y_\ell \in \mathcal{H}_\ell(\Omega)$ s.t. $\forall\ \hat{r} \in \Omega$

$$\left| f(\hat{r}) - \sum_{\ell=0}^{L} Y_\ell(\hat{r}) \right| < \varepsilon .$$

In words: any <u>continuous</u> fcn $f: \Omega \to \mathbb{C}$ can be approximated <u>uniformly</u> (ε is independent of $\hat{r}$) by a series of spherical harmonics.

Proof: any <u>poly</u> $P: \Omega \to \mathbb{C}$ can be written as a finite sum of $Y_\ell(\hat{r})$. Therefore if we can find a poly P s.t.

$$|f(\hat{r}) - P(\hat{r})| < \varepsilon$$

then we are done. But the existence of such a poly is exactly the assertion of the well-known:

<u>Weierstrass Approximation Theorem</u>: Suppose V is a bounded closed subset of R_3 (e.g. Ω) and $f: V \to C$ is continuous. Then $\forall\, \epsilon > 0$ $\exists$ a poly P s.t. $\forall\, \underline{r} \in V$

$$|f(\underline{r}) - P(\underline{r})| < \varepsilon .$$

i.e. any <u>cont.</u> fcn. can be uniformly approximated by a poly. on a bd, closed domain. For a proof (in 1-d, but easily extended to 3-d, see pp. 65-68 of Courant + Hilbert, vol. 1).

23. <u>The structure of the space $\mathcal{H}_\ell$</u>

We can now find its <u>dimension</u>

$$\overset{\mathcal{H}_\ell}{\mathcal{P}_\ell} = \overset{Y_\ell}{\mathcal{H}_\ell} \oplus r^2 \overset{Y_{\ell-2}}{\mathcal{H}_{\ell-2}} \oplus \cdots$$

$$= \mathcal{H}_\ell \oplus r^2 \underbrace{[\mathcal{H}_{\ell-2} \oplus \cdots]}_{\text{homogeneous of degree } \ell-2}$$

$$= \overset{Y_\ell}{\cancel{\mathcal{H}_\ell}} \oplus r^2 \overset{Y_{\ell-2}}{\cancel{\mathcal{P}_{\ell-2}}}$$

Thus $\boxed{\dim \mathcal{P}_\ell = \dim \mathcal{H}_\ell + \dim \mathcal{P}_{\ell-2}}$

But $\dim \mathcal{P}_\ell = \dfrac{(\ell+1)(\ell+2)}{2}$, so

$(\ell+1)(\ell+2)/2 = \dim \mathcal{H}_\ell + \ell(\ell-1)/2$

Thus $\boxed{\dim \mathcal{H}_\ell = \dim \mathcal{H}_\ell(\Omega) = 2\ell+1}$

There is a basis of each space $\mathcal{H}_\ell$ consisting of $2\ell+1$ lin. ind. solid spherical harmonics, and a corresponding basis of $\mathcal{H}_\ell(\Omega)$ consisting of $2\ell+1$ lin. ind. surface spherical harmonics.

Examples: (i) $\ell = 0$ $\dim \mathcal{H}_0 = \dim \mathcal{H}_0(\Omega) = 1$
$\mathcal{H}_0$ consists of the space of all complex constants. The canonical or conventional basis for $\mathcal{H}_\ell(\Omega)$ is orthonormal, elements denoted by $Y_\ell^m(\hat{r})$, $-\ell \le m \le \ell$, satisfy:

$$\boxed{(Y_\ell^m, Y_\ell^{m'}) = \int_\Omega Y_\ell^{m*} Y_\ell^{m'}\, dA = \delta_{mm'}}$$

Corresponding canonical basis for $\mathcal{H}_\ell$ has elements

$$\boxed{\mathcal{Y}_\ell^m(\underline{r}) = r^\ell\, Y_\ell^m(\hat{r})}$$

Thus $Y_0^0(\hat{r}) = \mathcal{Y}_0^0(\underline{r}) = (1/4\pi)^{1/2}$, since that renders
$\|Y_0^0\| = (Y_0^0, Y_0^0)^{1/2} = 1$.

(ii) $\ell = 1$, $\dim \mathcal{H}_1 = \dim H_1(\Omega) = 3$
A possible basis for $\mathcal{H}_1$ consists of x, y and z, since they are lin. ind.
Note that $\mathcal{P}_1 \equiv \mathcal{H}_1$, i.e. every poly of degree 1 is harmonic.

x, y, z is however not the canonical basis. It is in fact

$$\mathcal{Y}_1^0(\underline{r}) = (3/4\pi)^{1/2}\, z$$
$$\mathcal{Y}_1^{\pm 1}(\underline{r}) = \mp (3/8\pi)^{1/2}\,(x \pm iy)$$

and thus since $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$$Y_1^0(\hat{r}) = (3/4\pi)^{1/2}\cos\theta$$
$$Y_1^{\pm 1}(\hat{r}) = \mp (3/8\pi)^{1/2}\sin\theta\, e^{\pm i\phi}$$

There is a straightforward procedure for constructing the canonical orthonormal basis for $\mathcal{H}_\ell$ and $\mathcal{H}_\ell(\Omega)$ in each CAS $\hat{x}, \hat{y}, \hat{z}$.

If all we want is an orthonormal basis, could use Gram-Schmidt on any lin. ind. set of $2\ell+1$ vectors.

We shall however follow the procedure which is standard in quantum mechanics. First a definition.

Defn: Let V be a real (complex) vector space and let $T: V \to V$ be a linear operator. Let λ be a real (complex) number. Then $v \in V$ is said to be an eigenvector of T with associated eigenvalue λ if

(i) $v \neq 0$, and
(ii) $Tv = \lambda v$

Note: if v is an eigenvector, so is av, where a is any real (complex) non-zero scalar. Thus every eigenvalue λ has an associated eigenvector $\hat{n}$ of unit length $\|\hat{n}\| = 1$.

$$T\hat{n} = \lambda\hat{n}, \quad \|n\| = 1.$$

Example: $(x+iy)^\ell$ and $(x-iy)^\ell$ are both in $\mathcal{H}_\ell$, since

$$\nabla^2 (x \pm iy)^\ell = (\partial_x^2 + \partial_y^2 + \partial_z^2)(x \pm iy)^\ell$$
$$= \ell(\ell-1)\,[\,1 + i^2\,]\,(x + iy)^{\ell-2} = 0.$$

Now $(x \pm iy)^\ell = (r\sin\theta\, e^{\pm i\phi})^\ell$

$$= r^\ell (\sin\theta)^\ell\, e^{\pm i\ell\phi}.$$

Now recall $L_z = \frac{1}{i}\Lambda_z = \frac{1}{i}\partial_\phi$, so

$$\boxed{\begin{aligned} L_z\,(x+iy)^\ell &= \ell\,(x+iy)^\ell \\ L_z\,(x-iy)^\ell &= -\ell\,(x-iy)^\ell \end{aligned}}$$

Thus both $(x \pm iy)^\ell$ are eigenvectors of the operator $L_z : \mathcal{H}_\ell \to \mathcal{H}_\ell$. The associated eigenvalues are $\pm \ell$.

We are going to make $(x-iy)^\ell$ the first element of our canonical o.n. basis for $\mathcal{H}_\ell$. Let us thus normalize it.

$$\left((x-iy)^\ell, (x-iy)^\ell\right) = \int_\Omega |x-iy|^{2\ell}\, dA$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, (\sin\theta)^{2\ell}$$

$$= 2\pi \int_0^{\pi} (\sin\theta)^{2\ell+1} \, d\theta$$

$$= 4\pi \, \frac{2^{2\ell} (\ell!)^2}{(2\ell+1)!}$$

Thus define

$$\mathcal{Y}_\ell^{-\ell} \equiv \sqrt{\frac{2\ell+1}{4\pi}} \left[\frac{(2\ell)!}{0!} \right]^{1/2} \frac{r^\ell}{2^\ell \ell!} \sin^\ell\theta \, e^{-i\ell\phi}$$

$$\mathcal{Y}_\ell^{-\ell}(\underline{r}) = r^\ell \, Y_\ell^{-\ell}(\hat{r})$$

$$Y_\ell^{-\ell} \equiv \sqrt{\frac{2\ell+1}{4\pi}} \left[\frac{(2\ell)!}{0!} \right]^{1/2} \frac{1}{2^\ell \ell!} \sin^\ell\theta \, e^{-i\ell\phi}$$

Then $Y_\ell^{-\ell} \in H_\ell(\Omega)$ is of unit length $\| Y_\ell^{-\ell} \|^2 = (Y_\ell^{-\ell}, Y_\ell^{-\ell}) = 1$. So is $\mathcal{Y}_\ell \in \mathcal{H}_\ell$.

We will construct a basis starting with $Y_\ell^{-\ell}$ using as a tool the L_+ <u>ladder operator</u>. The property of L_+ of use to us is this: Suppose that

$$L_z H_\ell^m = m H_\ell^m \quad \text{and} \quad H_\ell^m \in \mathcal{H}_\ell .$$

Then

$$L_+ H_\ell^m \equiv H_\ell^{m+1}$$ is also $\in \mathcal{H}_\ell$ and

$$L_z H_\ell^{m+1} = (m+1) H_\ell^{m+1}.$$

Proof: $L_z H_\ell^{m+1} = L_z L_+ H_\ell^m = L_+ (L_z+1) H_\ell^m$

(since $[L_z, L_+] = L_+$)

$$= L_+ (m+1) H_\ell^m = (m+1) L_+ H_\ell^m$$

$$= (m+1) H_\ell^{m+1}$$

Note that we have not shown that H_ℓ^{m+1} is an eigenfunction of L_z with eigenvalue $m+1$ since we have not investigated whether or not $H_\ell^{m+1} = 0$.

Is H_ℓ^{m+1} an eigenvector of L_z, i.e. is it $\neq 0$

$$(H_\ell^{m+1}, H_\ell^{m+1}) = (L_+ H_\ell^m, L_+ H_\ell^m)$$

$$= (H_\ell^m, L_- L_+ H_\ell^m) = (H_\ell^m, (L^2 - L_z^2 - L_z) H_\ell^m)$$

$$= [\ell(\ell+1) - m^2 - m] (H_\ell^m, H_\ell^m)$$

$$= (\ell-m)(\ell+m+1) (H_\ell^m, H_\ell^m)$$

Thus $H_\ell^{m+1} \neq 0$ provided $H_\ell^{m} \neq 0$
if $m \neq -\ell-1$ and $m \neq \ell$.
Furthermore H_ℓ^{m} must vanish for $m > \ell$.

Thus, over a restricted range of m, L_+ has the property that if H_ℓ^{m} is an eigenvector of $L_z : H_\ell \to H_\ell$ with eiv m, then $L_+ H_\ell^{m} = H_\ell^{m+1}$ is also an eigenvector of L_z with eiv $m+1$.
Reason for name: <u>ladder operator</u>.

<u>Theorem</u>: main theorem on structure of <u>$H_\ell(\Omega)$</u> and H_ℓ. <u>Define $Y_\ell^{-\ell}$ as above and define $Y_\ell^{m} = 0$ if $|m| > \ell$. Also define</u>

$$\boxed{\begin{gathered} Y_\ell^{m} \equiv \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} \frac{1}{\sqrt{(2\ell)!}} \, (L_+)^{\ell+m} \, Y_\ell^{-\ell}, \\ -\ell \leq m \leq \ell \end{gathered}}$$

Then $Y_\ell^{-\ell}, \dots Y_\ell^{0}, \dots Y_\ell^{\ell}$ are an orthonormal basis for $H_\ell(\Omega)$ and furthermore, the following formulae are true <u>for all m</u>.

$$\boxed{\begin{array}{ll} 1. & L_+ Y_\ell^m = \sqrt{(\ell-m)(\ell+m+1)}\; Y_\ell^{m+1} \\ 2. & L_- Y_\ell^m = \sqrt{(\ell+m)(\ell-m+1)}\; Y_\ell^{m-1} \\ 3. & L_z Y_\ell^m = m\, Y_\ell^m \end{array}}$$

Proof: prove the formulae 1.2.3. first.

1. for $m \le -\ell-2$: trivial (all zero)

for $m = -\ell-1$: trivial ($\ell+m+1 = 0$)

for $-\ell \le m \le \ell-1$;

$$L_+ Y_\ell^m = \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} \frac{1}{\sqrt{(2\ell)!}} (L_+)^{\ell+m+1} Y_\ell^{-\ell}$$

$$= \sqrt{(\ell-m)(\ell+m+1)}\; Y_\ell^{m+1}.$$

for $m = \ell$:

$$(L_+ Y_\ell^\ell, L_+ Y_\ell^\ell) = (Y_\ell^\ell, L_- L_+ Y_\ell^\ell)$$
$$= (Y_\ell^\ell, (L^2 - L_z^2 - L_z) Y_\ell^\ell)$$
$$= [\ell(\ell+1) - \ell^2 - \ell]\,(Y_\ell^\ell, Y_\ell^\ell) = 0$$

for $m > \ell$: trivial

2. apply L_- to 1. to get

$$(L^2 - L_z^2 - L_z) Y_\ell^m = \sqrt{(\ell-m)(\ell+m+1)}\; L_- Y_\ell^{m+1}$$

(this uses 3., which should be proved before 2.)

$$(l-m)(l+m+1)\,Y_l^m = \sqrt{(l-m)(l+m+1)}\;L_-Y_l^{m+1}$$

or $$L_-Y_l^{m+1} = \sqrt{(l-m)(l+m+1)}\;Y_l^m$$

change m to $m-1$ (can do since 1. true $\forall m$).

$$L_-Y_l^m = \sqrt{(l+m)(l-m+1)}\;Y_l^{m-1}$$

3. prove by <u>induction</u> on m.
~~[illegible]~~
trivial for $|m| > l$ (all zero)
true for $m = -l$.
<u>Now</u> assume true for m. Then
true for $m+1$ because

$$L_z Y_l^{m+1} = L_z \frac{L_+Y_l^m}{\sqrt{(l-m)(l+m+1)}} = \frac{L_+(L_z+1)Y_l^m}{\sqrt{(l-m)(l+m+1)}}$$

$$= \frac{(m+1)\,L_+Y_l^m}{\sqrt{(l-m)(l+m+1)}} = (m+1)\,Y_l^{m+1}.$$

Note induction fails if $m = l$.

Now we still must prove the Y_l^m are <u>orthonormal</u> and <u>lin. ind.</u> This is easy now that we have 1.2.3. at our disposal.

That $\| Y_\ell^m \| = 1$ is also proved by induction on m. True for $m=-\ell$. Suppose true, i.e. $\| Y_\ell^m \| = 1$ for m satisfying $-\ell \leq m \leq \ell-1$. Then true for $m+1$ since

$$(Y_\ell^{m+1}, Y_\ell^{m+1}) = \frac{1}{(\ell-m)(\ell+m+1)} (L_+ Y_\ell^m, L_+ Y_\ell^m)$$

$$= \frac{1}{(\ell-m)(\ell+m+1)} (Y_\ell^m, L_- L_+ Y_\ell^m)$$

$$= \frac{1}{(\ell-m)(\ell+m+1)} (Y_\ell^m, (L^2 - L_z^2 - L_z) Y_\ell^m)$$

$$= (Y_\ell^m, Y_\ell^m) = 1.$$

Again induction fails (as expected) at $m=\ell$.

Now to prove <u>lin. ind.</u> But Y_ℓ^m is an eigenvector of the operator $L_z : \mathcal{H}_\ell(\Omega) \to \mathcal{H}_\ell(\Omega)$, with eiv m. (when $|m| \leq \ell$). Furthermore L_z is Hermitian. We may thus call on the well-known <u>theorem</u>

<u>Theorem</u>: If λ_1 and λ_2 are two <u>distinct</u> eigenvalues of a <u>Hermitian</u> operator $L : V \to V$

with associated eigenvectors v_1 and v_2, then $(v_1, v_2) = 0$.

i.e. eigenvectors assoc. with distinct eigenvalues are <u>orthogonal</u>. The proof is simple.

$$Lv_1 = \lambda_1 v_1 \quad , \quad Lv_2 = \lambda_2 v_2$$

$$(v_2, Lv_1) = (v_2, \lambda_1 v_1)$$

$$= \lambda_1^* (v_2, v_1) \qquad \text{but}$$

$$(v_2, Lv_1) = (Lv_2, v_1) \qquad (\underline{\text{Hermitian}})$$

$$= (v_1, Lv_2)^* = (v_1, \lambda_2 v_2)^*$$

$$= \lambda_2 (v_1, v_2)^* = \lambda_2 (v_2, v_1)$$

Thus $\lambda_1^* (v_2, v_1) = \lambda_2 (v_2, v_1)$ or

$$\boxed{(\lambda_2 - \lambda_1^*)(v_2, v_1) = 0 \quad *}$$

Can now prove two things:

(i) take $1 \equiv 2$, then $(v_1, v_1) \neq 0$ if v_1 is an eigenvector, so $\lambda_1^* - \lambda_1 = 0$ i.e. λ_1 is <u>real</u>, i.e. <u>all</u> λ_i are <u>real</u>.

(ii) can now write $*$ $(\lambda_2 - \lambda_1)(v_2, v_1) = 0$. take $\lambda_2 \neq \lambda_1$ (<u>distinct</u> eivs). Then $(v_2, v_1) = 0$, i.e. eigenvectors assoc. with distinct eivs are $\perp$.

Thus, since $L_z : \mathcal{H}_\ell(\Omega) \to \mathcal{H}_\ell(\Omega)$ is Hermitian, the Y_ℓ^m, $|m| \leq \ell$ are <u>mutually orthogonal</u>. Therefore they are <u>lin. ind</u>. Since there are $2\ell+1$ of them, they form a basis. Thus

$$\boxed{\begin{array}{l} Y_\ell^{-\ell}, \dots Y_\ell^0, \dots Y_\ell^\ell \text{ as defined above} \\ \text{constitute an orthonormal basis for} \\ \mathcal{H}_\ell(\Omega). \end{array}}$$

q.e.d.

Corollary: we can also express Y_ℓ^m in terms of the L_- ladder operator. Claim

$$\boxed{\begin{array}{c} Y_\ell^m = \left[\dfrac{(\ell+m)!}{(\ell-m)!}\right]^{1/2} \dfrac{1}{\sqrt{(2\ell)!}} \,(L_-)^{\ell-m}\, Y_\ell^\ell \,, \\ -\ell \leq m \leq \ell \end{array}}$$

Proof: induction again. True if $m=\ell$. Suppose true for m. Then

$$Y_\ell^{m-1} = \frac{1}{\sqrt{(\ell+m)(\ell-m+1)}}\, L_- Y_\ell^m = \left[\frac{(\ell+m-1)!}{(\ell-m+1)!}\right]^{1/2} \frac{1}{\sqrt{(2\ell)!}}$$

$$(L_-)^{\ell-m+1}\, Y_\ell^{\ell}$$

L_- is a descending ladder operator just as L_+ is an ascending ladder operator. We could have constructed the same basis by starting with $Y_\ell^\ell \sim (x+iy)^\ell$, normalize, and ladder downward to $Y_\ell^{-\ell}$.

The canonical basis $Y_\ell^{-\ell}, \dots Y_\ell^0, \dots Y_\ell^\ell$ of $H_\ell(\Omega)$ consists of the simultaneous eigenvectors of L^2 and L_z, normalized s.t. $\| Y_\ell^m \| = 1$. The fact that L^2 and L_z commute, i.e. $[L^2, L_z] = 0$, guarantees that these simultaneous eigenvectors exist, by virtue of the

Theorem: If V is a finite-dimensional inner product space and $S: V \to V$ and $T: V \to V$ are both Hermitian linear operators, then $\exists$ an orthonormal basis of V consisting of simultaneous eigenvectors of both S and T iff S and T commute, i.e. $ST = TS$.

This is the theorem which lies at the ♡ of quantum mechanics. The proof of this is simple. Let's just sketch it.

First if there is such an o.n. basis, then the matrix of components of T and S in that basis have the form $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ and $\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$

where λ_i, σ_i are the eigenvalues of T, S respectively. Clearly these two matrices commute. Thus T and S commute.

Suppose now that $ST = TS$. Let $\{T, \lambda\}$ denote the subspace of V consisting of eigenvectors of T with assoc. eiv. λ. Then ~~[illegible]~~ if $v \in \{T, \lambda\}$, $Tv = \lambda v$. Then

$$TSv = STv = S\lambda v = \lambda S v$$
$$T(Sv) = \lambda (Sv)$$

Thus $Sv \in \{T, \lambda\}$. The ~~[illegible]~~ operator defined by <u>restricting</u> S to $\{T, \lambda\}$ is clearly Hermitian, so we know $\exists$ an ~~[illegible]~~ o.n. basis of $\{T, \lambda\}$ consisting of eigenvectors of S. Being in $\{T, \lambda\}$ these vectors are also eigenvectors of T. The same construction can clearly be repeated for <u>each</u> eigenspace of T. The result is an o.n. basis of V consisting of simultaneous eigenvectors of S and T.

The q.m. significance of this is well-known. ~~[illegible]~~ A complete ~~[illegible]~~ description of a q.m. system consists of a knowledge of L^2 and L_z. They are said to constitute a complete set of commuting observables. A stationary angular momentum state is defined by the two quantum numbers $l(l+1)$ and m, the eigenvalues of L^2 and L_z.

Actually, in q.m. because of spin, both l and m are allowed to assume half-integral values.

The ladder operators $L_\pm$ convert an eigenvector with eigenvalue (of L_z) m into an eigenvector with eigenvalue $m \pm 1$. Hence the name.

24. Explicit expressions for the Y_ℓ^m:

We have defined, for $|m| \le \ell$,

$$Y_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}}\ \frac{1}{2^\ell \ell!}\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} (L_+)^{\ell+m} \sin^\ell\theta\, e^{-i\ell\phi}$$

where $L_+ = e^{i\phi}\left(\partial_\theta + i\cot\theta\, \partial_\phi\right)$

Now for any fcn $f(\theta)$ only:

$$L_+ e^{iq\phi} f(\theta) = e^{i(q+1)\phi}\left(\partial_\theta - q\cot\theta\right) f(\theta)$$

$$= e^{i(q+1)\phi} \sin^q\theta\ \partial_\theta \sin^{-q}\theta\, f(\theta)$$

$$(L^+)^2 e^{iq\phi} f(\theta) = e^{i(q+2)\phi}\left[\partial_\theta - (q+1)\cot\theta\right]$$

$$\sin^q\theta\ \partial_\theta \sin^{-q}\theta\ f(\theta)$$

$$= e^{i(q+2)\phi} \sin^{q+1}\theta\ \partial_\theta\left(\frac{1}{\sin\theta}\right)\partial_\theta \sin^{-q}\theta\ f(\theta)$$

$$= e^{i(q+2)\phi} \sin^{q+2}\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)^2 \sin^{-q}\theta\ f(\theta)$$

In general

$$(L_+)^\nu e^{iq\phi} f(\theta) = e^{i(q+\nu)\phi} \sin^{q+\nu}\theta \left(\frac{1}{\sin\theta}\partial_\theta\right)^\nu \sin^{-q}\theta f(\theta)$$

Now set $\nu = l+m$, $q = -l$, $f(\theta) = \sin^l\theta$

Find:

$$Y_l^m = \sqrt{\frac{2l+1}{4\pi}} \frac{1}{2^l l!} \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} e^{im\phi} \sin^m\theta \left(\frac{1}{\sin\theta}\partial_\theta\right)^{l+m} \sin^{2l}\theta, \quad |m| \le l$$

Now let $\mu = \cos\theta$.

The <u>Legendre polynomial of degree l</u> can be defined by the <u>Rodrigues formula</u>:

$$\boxed{P_l(\mu) \equiv \frac{1}{2^l l!}\left(\frac{d}{d\mu}\right)^l (\mu^2-1)^l}$$

The <u>associated Legendre function of degree l and order m</u> is defined similarly by:

$$\boxed{P_\ell^m(\mu) \equiv \frac{1}{2^\ell \ell!}\,(1-\mu^2)^{m/2}\left(\frac{d}{d\mu}\right)^{\ell+m}(\mu^2-1)^\ell, \quad \text{valid for } |m|\leq \ell.}$$

Note that $P_\ell^0(\mu) = P_\ell(\mu)$.

Making use of $\partial_\mu = -\frac{1}{\sin\theta}\partial_\theta$, we can write our Y_ℓ^m in the form

$$\boxed{Y_\ell^m(\theta,\phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}}\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)\, e^{im\phi}, \quad |m|\leq \ell}$$

In particular we have two simple cases

$$Y_\ell^0 = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

$$Y_\ell^\ell = (-1)^\ell \sqrt{\frac{2\ell+1}{4\pi}}\,\frac{1}{2^\ell \ell!}\left[\frac{(2\ell)!}{0!}\right]^{1/2} \sin^\ell\theta\, e^{i\ell\phi}$$

Recall we started with

$$Y_\ell^{-\ell} = \sqrt{\frac{2\ell+1}{4\pi}}\,\frac{1}{2^\ell \ell!}\left[\frac{(2\ell)!}{0!}\right]^{1/2} \sin^\ell\theta\, e^{-i\ell\phi}$$

Note that $Y_\ell^{-\ell} = (-1)^\ell\, Y_\ell^{\ell\,*}$.

This is in fact more general, as we now show.

We also derived an expression for Y_ℓ^m in terms of L_-, viz.

$$Y_\ell^m = (-1)^\ell \sqrt{\frac{2\ell+1}{4\pi}}\, \frac{1}{2^\ell \ell!} \left[\frac{(\ell+m)!}{(\ell-m)!}\right]^{1/2} (L_-)^{\ell-m}\, e^{i\ell\phi} \sin^\ell\theta$$

$$L_- = e^{-i\phi}\left[-\partial_\theta + i\cot\theta\, \partial_\phi\right]$$

Consider:

$$\begin{aligned} L_-\, e^{iq\phi} f(\theta) &= -e^{i(q-1)\phi} (\partial_\theta + q\cot\theta) f(\theta) \\ &= -e^{i(q-1)\phi} \sin^{-q}\theta\, \partial_\theta \sin^q\theta\, f(\theta) \end{aligned}$$

$$\begin{aligned} (L_-)^2 e^{iq\phi} f(\theta) &= (-1)^2 e^{i(q-2)\phi} \sin^{-q+1}\theta\, \partial_\theta \frac{1}{\sin\theta} \partial_\theta \sin^q\theta\, f(\theta) \\ &= e^{i(q-2)\phi} \sin^{-q+2}\theta \left(-\frac{1}{\sin\theta}\partial_\theta\right)^2 \sin^q\theta\, f(\theta) \end{aligned}$$

In general

$$(L_-)^\nu e^{iq\phi} f(\theta) = e^{i(q-\nu)\phi} \sin^{-q+\nu}\theta \left(-\frac{1}{\sin\theta}\partial_\theta\right)^\nu \sin^q\theta\, f(\theta)$$

Now set $\nu = \ell - m$, $q = \ell$, $f(\theta) = \sin^\ell\theta$.

$$\boxed{Y_\ell^m = (-1)^\ell \sqrt{\frac{2\ell+1}{4\pi}}\, \frac{1}{2^\ell \ell!} \left[\frac{(\ell+m)!}{(\ell-m)!}\right]^{1/2} e^{im\phi} \sin^{-m}\theta \left(-\frac{1}{\sin\theta}\partial_\theta\right)^{\ell-m} \sin^{2\ell}\theta\, , \quad |m| \le \ell}$$

Comparing this with $*$, we see that

$$\boxed{Y_\ell^{-m} = (-1)^m Y_\ell^{m\,*}} \quad *$$

<u>Let's see now what the Y_ℓ^m look like.</u>

$Y_\ell^m(-\hat{r}) = (-1)^\ell Y_\ell^m(\hat{r})$ since Y_ℓ^m is <u>homogeneous of degree ℓ</u>

in the language of q.m. one says the parity of Y_ℓ^m is equal to its degree

$$\boxed{Y_\ell^m(-\hat{r}) = (-1)^\ell Y_\ell^m(\hat{r})} \quad **$$

It follows that $P_\ell^m(-\mu) = (-1)^{\ell+m} P_\ell^m(\mu)$.

Let us write $\boxed{Y_\ell^m(\theta,\phi) = X_\ell^m(\theta)\, e^{im\phi}}$

Then $X_\ell^m(\theta)$ is real and from $*$ and $**$ satisfies

$$\boxed{\begin{aligned} X_\ell^{-m}(\theta) &= (-1)^m X_\ell^m(\theta) \\ X_\ell^m(\pi-\theta) &= (-1)^{\ell+m}(\theta) \end{aligned}} \quad ***$$

If $X_\ell^m(\theta)$ is known for positive m, it can be found for negative m from ***

Note that if:

$\ell + m$ even: X_ℓ^m symmetric across equator

$\ell + m$ odd: X_ℓ^m anti-symmetric across equator, vanishes at equator $\theta = \pi/2$.

Now consider near the pole: $\theta = 0$.
We make use of

$$X_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{2^\ell \ell!} \left[\frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2} \sin^m\theta$$

$$\left(\frac{1}{\sin\theta} \partial_\theta \right)^{\ell+m} \sin^{2\ell}\theta$$

Near the pole $\sin\theta \sim \theta$

Consider:

$$\theta^m \left(\frac{1}{\theta} \partial_\theta \right)^{\ell+m} \theta^{2\ell}$$

$m = -\ell$: $\quad \theta^\ell$

$m = -(\ell-1)$: $\quad 2\ell\, \theta^{\ell-1}$

$m = -(\ell-2)$: $\quad 2\ell\,(2\ell-2)\,\theta^{\ell-2} = 2^2 \ell(\ell-1)\,\theta^{\ell-2}$

$\vdots$

$m = 0$: $\quad 2^\ell \ell!\, \theta^0$

Thus for $-\ell \le m \le 0$, we find

$$X_\ell^m(\theta) \sim \sqrt{\frac{2\ell+1}{4\pi}} \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} \frac{1}{2^{|m|}\,|m|!}\ \theta^{|m|}$$

For $0 < m \leq \ell$, the above argument does not work, and we make use instead of

$$X_\ell^m = (-1)^\ell \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{2^\ell \ell!} \left[\frac{(\ell+m)!}{(\ell-m)!}\right]^{1/2}$$

$$\sin^{-m}\theta \left(-\frac{1}{\sin\theta}\partial_\theta\right)^{\ell-m} \sin^{2\ell}\theta$$

Once again for $\theta \sim 0$, $\sin\theta \sim \theta$. We consider

$$(-1)^\ell \theta^{-m} \left(-\frac{1}{\theta}\partial_\theta\right)^{\ell-m} \theta^{2\ell}$$

$m = \ell$: $\quad (-1)^\ell \theta^\ell$

$m = \ell-1$: $\quad (-1)^{\ell-1}\, 2\ell\, \theta^{\ell-1}$

$\vdots$

$m = 0$: $\quad (-1)^0\, 2^\ell\, \ell!\, \theta^0$

Thus for $0 \leq m \leq \ell$, we find

$$X_\ell^m(\theta) \sim (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} \left[\frac{(\ell+m)!}{(\ell-m)!}\right]^{1/2} \frac{1}{2^m\, m!}\, \theta^m\, e^{im\phi}$$

In summary, if we <u>define</u>

$$b_m \equiv \frac{(-1)^m}{2^{|m|}\,|m|!} \sqrt{\frac{2\ell+1}{4\pi}} \left[\frac{(\ell+|m|)!}{(\ell-|m|)!}\right]^{1/2}$$

Then near the pole i.e. $\theta \sim 0$,

$$\boxed{\begin{aligned} X_\ell^m(\theta) &\sim b_m\, \theta^m \,, \qquad 0 \le m \le \ell \\ X_\ell^m(\theta) &\sim (-1)^m b_m\, \theta^{|m|}, \quad -\ell \le m \le 0 \end{aligned}} \;\; *$$

The formulae $*$ are very useful in normal mode excitation calculations. Note that, as expected, $X_\ell^{-m}(\theta) = (-1)^m X_\ell^m(\theta)$ is conserved by $*$.

Right at the poles $\hat{r} = \pm \hat{z}$:

$$\boxed{\begin{aligned} &Y_\ell^0(\hat{z}) = \sqrt{\tfrac{2\ell+1}{4\pi}} \qquad Y_\ell^0(-\hat{z}) = (-1)^\ell \sqrt{\tfrac{2\ell+1}{4\pi}} \\ &Y_\ell^m(\hat{z}) = Y_\ell^m(-\hat{z}) = 0 \quad \text{for } m \neq 0 \end{aligned}}$$

What do plots of $X_\ell^m(\theta)$ look like?
For $m = 0$,

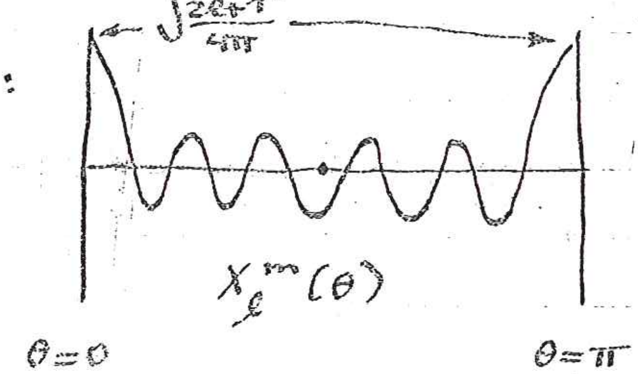
$$X_\ell^0(\theta) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

$*$ $P_\ell(\mu) = \dfrac{1}{2^\ell \ell!} \left(\dfrac{d}{d\mu}\right)^\ell (\mu^2-1)^\ell$ is a polynomial of degree ℓ in μ.

It has ℓ zeroes, all located between $\mu = \pm 1$.

Thus $X_\ell^0(\theta)$ has ℓ zeroes or nodes between $\theta = 0$ and $\theta = \pi$.

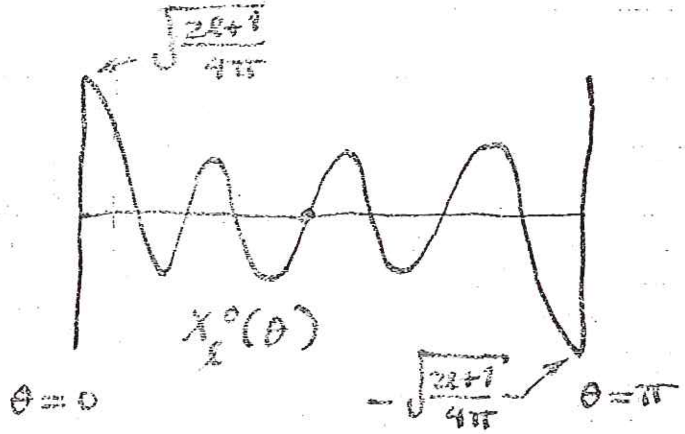
<u>ℓ even</u>:



ℓ nodes, symmetric or even across equator since

$$X_\ell^m(\pi - \theta) = (-1)^m X_\ell^m(\theta)$$

<u>ℓ odd</u>:



ℓ nodes, but antisymmetric or odd across equator. Thus one node at <u>equator</u>

At the equator: $X_\ell^0(\pi/2) = 0$, ℓ odd

Can be shown that for <u>ℓ even</u>

$$X_\ell^0(\pi/2) = \sqrt{\frac{2\ell+1}{4\pi}} \; (-1)^{\ell/2} \frac{\ell!}{2^\ell \left[(\ell/2)!\right]^2}$$

For <u>large</u> even ℓ, from Stirling's formula we then find

$$X_\ell^0(\pi/2) \sim (-1)^{\ell/2} \frac{1}{\pi}$$

Now for $m \neq 0$:

$$P_\ell^m(\mu) = \frac{(1-\mu^2)^{m/2}}{2^\ell \ell!} \left(\frac{d}{d\mu}\right)^{\ell+m} (\mu^2-1)^\ell \qquad *$$

This is valid for negative as well as positive m. For $m \geq 0$, we can combine * above with * on page 202 to get

$$\boxed{P_\ell^m(\mu) = (1-\mu^2)^{m/2} \left(\frac{d}{d\mu}\right)^m P_\ell(\mu), \quad 0 \leq m \leq \ell.}$$

This shows that for $0 \leq m \leq \ell$, $X_\ell^m(\theta)$ has $\ell - m$ nodes between $\theta = 0$ and π. Thus since

$$\boxed{X_\ell^{-m}(\theta) = (-1)^m X_\ell^m(\theta)}$$

$\boxed{X_\ell^m(\theta) \text{ has } \ell - |m| \text{ nodes between } \theta = 0 \text{ and } \theta = \pi.}$

ℓ even, m even or ℓ odd, m odd: X_ℓ^m <u>even</u> across equator.

ℓ even, m odd or ℓ odd, m even: X_ℓ^m odd across equator.

Table 1. Harmonic Polynomials and Spherical Harmonics.

$$\mathcal{Y}_{lm}(\mathbf{r}) = r^l Y_{lm}(\theta, \varphi)$$

l, m	$\mathcal{Y}_{lm}(\mathbf{r})$	$Y_{lm}(\theta, \varphi)$
0 0	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{2\sqrt{\pi}}$
1 0	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\, z$	$\frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$
1 ±1	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\,(x \pm iy)$	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta e^{\pm i\varphi}$
2 0	$\frac{1}{4}\sqrt{\frac{5}{\pi}}\,(2z^2 - x^2 - y^2)$	$\frac{1}{4}\sqrt{\frac{5}{\pi}}\,(2\cos^2\theta - \sin^2\theta)$
2 ±1	$\mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\, z(x \pm iy)$	$\mp\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cos\theta \sin\theta e^{\pm i\varphi}$
2 ±2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\,(x \pm iy)^2$	$\frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{\pm 2i\varphi}$
3 0	$\frac{1}{4}\sqrt{\frac{7}{\pi}}\,(2z^2 - 3x^2 - 3y^2)z$	$\frac{1}{4}\sqrt{\frac{7}{\pi}}\,(2\cos^3\theta - 3\cos\theta\sin^2\theta)$
3 ±1	$\mp\frac{1}{8}\sqrt{\frac{21}{\pi}}\,(4z^2 - x^2 - y^2)(x \pm iy)$	$\mp\frac{1}{8}\sqrt{\frac{21}{\pi}}\,(4\cos^2\theta\sin\theta - \sin^3\theta)e^{\pm i\varphi}$
3 ±2	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}\, z(x \pm iy)^2$	$\frac{1}{4}\sqrt{\frac{105}{2\pi}} \cos\theta\sin^2\theta e^{\pm 2i\varphi}$
3 ±3	$\mp\frac{1}{8}\sqrt{\frac{35}{\pi}}\,(x \pm iy)^3$	$\mp\frac{1}{8}\sqrt{\frac{35}{\pi}} \sin^3\theta e^{\pm 3i\varphi}$

Irreducible tensors containing in addition the components of some other vector $\mathbf{r}'$ may be constructed by polarization of the harmonics with the operator

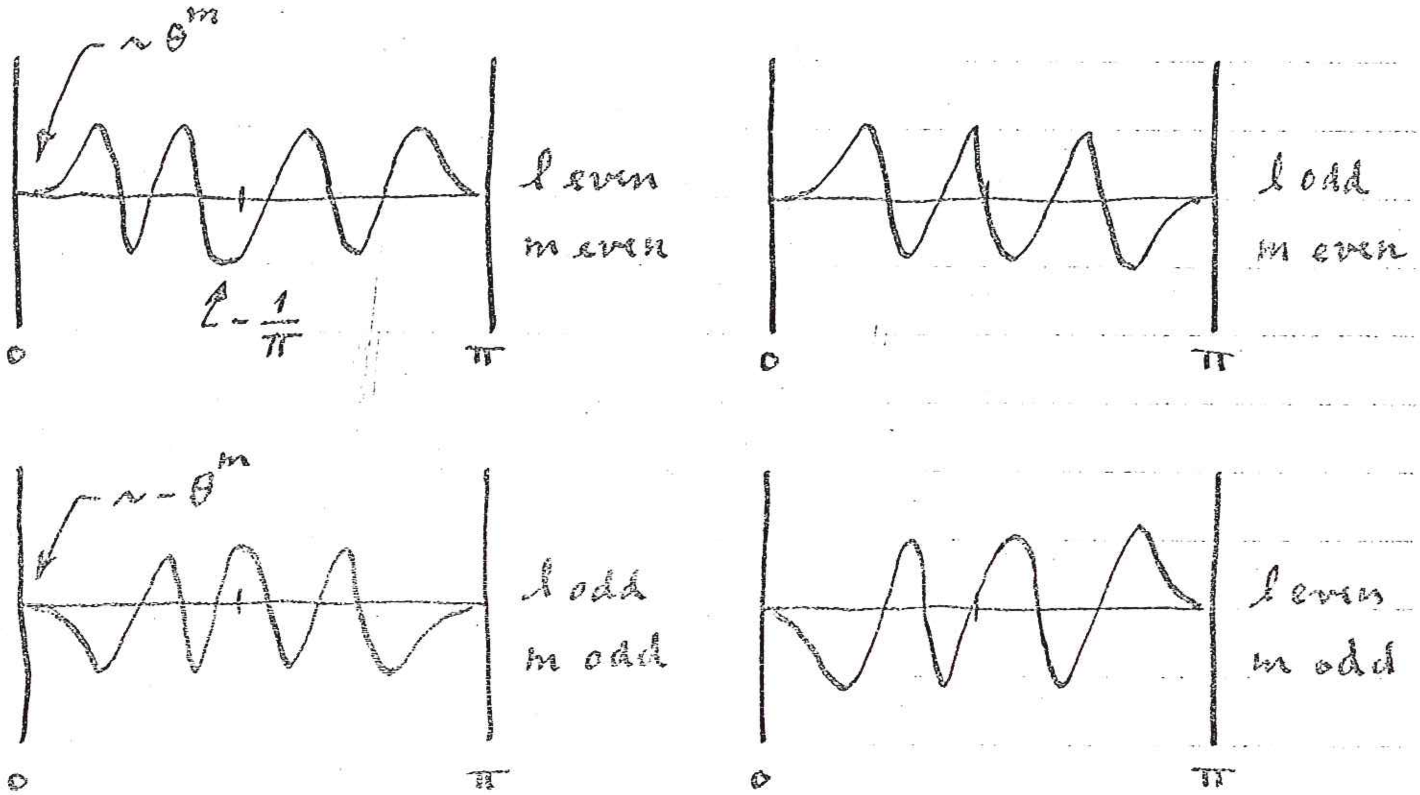
$$\mathbf{r}'\cdot\nabla \equiv x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}$$

Cf. Rose (1954).

Table 2.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\frac{1}{2}J}\left[\frac{(j_1+j_2-j_3)!(j_1+j_3-j_2)!(j_2+j_3-j_1)!}{(j_1+j_2+j_3+1)!}\right]^{\frac{1}{2}} \frac{(\frac{1}{2}J)!}{(\frac{1}{2}J-j_1)!(\frac{1}{2}J-j_2)!(\frac{1}{2}J-j_3)!}$$

If J is even.



both symmetric   both antisymmetric

<u>Edmonds</u> has a table giving $Y_\ell^m(\hat{r})$ and the associated $\mathcal{Y}_\ell^m(\underline{r})$ for l, m up to $3, \pm 3$.

<u>Orthogonality</u> of Y_ℓ^m : $(Y_\ell^m, Y_\ell^{m'}) = \delta_{mm'}$

or

$$\boxed{\int_\Omega Y_\ell^m \, Y_\ell^{m'*} \, dA = \delta_{mm'}}$$

We can use this to obtain orthogonality relations for the Legendre polys P_ℓ and the associated Legendre functions P_ℓ^m, viz.

$$\int_{-1}^{1} P_\ell(\mu)\, P_{\ell'}(\mu)\, d\mu = \frac{\delta_{\ell\ell'}}{\ell+1/2}$$

$$\int_{-1}^{1} P_\ell^m(\mu)\, P_{\ell'}^m(\mu)\, d\mu = \frac{\delta_{\ell\ell'}}{\ell+1/2} \left[\frac{(\ell+m)!}{(\ell-m)!} \right]$$

Note: we do not have

$$\int_{-1}^{1} P_\ell^m\, P_\ell^{m'}\, d\mu = 0, \quad m \neq m'.$$

26. Asymptotic formulae for Y_ℓ^m, $\ell \gg 1$.

The p.d.e. satisfied by Y_ℓ^m is

$$\Lambda^2 Y_\ell^m = -\ell(\ell+1) Y_\ell^m, \quad \text{or equivalently}$$

$$\nabla_1^2 Y_\ell^m = -\ell(\ell+1) Y_\ell^m \quad \text{where}$$

$$\nabla_1^2 = \Lambda^2 = \frac{1}{\sin\theta}\, \partial_\theta (\sin\theta\, \partial_\theta) + \frac{1}{\sin^2\theta}\, \partial_\phi^2$$

The o.d.e. satisfied by $X_\ell^m(\theta)$ is thus

$$\frac{1}{\sin\theta}\, \partial_\theta (\sin\theta\, \partial_\theta) X_\ell^m - \frac{m^2}{\sin^2\theta} X_\ell^m$$

$$+ \ell(\ell+1) X_\ell^m = 0, \quad \text{or}$$

$$\boxed{\begin{aligned}\partial_\theta^2 X_\ell^m &+ \cot\theta\, \partial_\theta X_\ell^m \\ &+ \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] X_\ell^m = 0\end{aligned}}$$

Make the substitution $\mu = \cos\theta$

$$\partial_\theta = -\sin\theta\, \partial_\mu$$
$$\partial_\theta^2 = -\cos\theta\, \partial_\mu + \sin^2\theta\, \partial_\mu^2$$

$$\partial_\theta = -\sqrt{1-\mu^2}\, \partial_\mu$$
$$\partial_\theta^2 = -\mu\, \partial_\mu + (1-\mu^2)\, \partial_\mu^2$$

$$\cot\theta\, \partial_\theta = -\mu\partial_\mu .$$

Thus find that $P_\ell^m(\mu)$ satisfies :

$$\boxed{\begin{aligned}(1-\mu^2)\frac{d^2y}{d\mu^2} &- 2\mu\frac{dy}{d\mu} \\ &+ \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right] y = 0\end{aligned}}$$

*

This is Legendre's diff. eqn.

Let us now change the dependent variable to :

$$\boxed{w = (1-\mu^2)^{1/2}\, u}$$

or $\quad y = (1-\mu^2)^{-1/2}\, w$

Let a $'$ denote $d/d\mu$. Then

$$y' = \mu(1-\mu^2)^{-3/2}\, w + (1-\mu^2)^{-1/2}\, w'$$

$$y'' = [\,(1-\mu^2)^{-3/2} + 3\mu^2 (1-\mu^2)^{-5/2}\,]\, w$$

$$+ 2\mu(1-\mu^2)^{-3/2}\, w' + (1-\mu^2)^{-1/2}\, w''$$

Thus $\quad (1-\mu^2)\, y'' - 2\mu\, y'$

$$= (1-\mu^2)^{1/2}\, w'' + \underline{2\mu\,(1-\mu^2)^{-1/2}\, w'}$$

$$+ [\,(1-\mu^2)^{-1/2} + 3\mu^2 (1-\mu^2)^{-3/2}\,]\, w$$

$$\underline{- 2\mu\,(1-\mu^2)^{-1/2}\, w'} - 2\mu^2 (1-\mu^2)^{-3/2}\, w$$

The $'$ term cancels out. We get from *

$$(1-\mu^2)^{1/2}\, w'' + [\,(1-\mu^2)^{-1/2} + \mu^2 (1-\mu^2)^{-3/2}$$

$$+ \ell(\ell+1)(1-\mu^2)^{-1/2} - m^2 (1-\mu^2)^{-3/2}\,]\, w = 0$$

$$w'' + [\,(\ell(\ell+1)+1)(1-\mu^2)^{-1} + (\mu^2 - m^2)$$

$$(1-\mu^2)^{-2}\,]\, w = 0$$

Let $\boxed{\varepsilon \equiv \dfrac{1}{\sqrt{l(l+1)}} \quad \text{and} \quad a = \sqrt{1 - \dfrac{m^2}{l(l+1)}}}$

Then Legendre's eqn. becomes

$$\boxed{\varepsilon^2 w'' + \left[\frac{a^2 - \mu^2 + \varepsilon^2}{(1-\mu^2)^2}\right] w = 0 \quad *}$$

For $\varepsilon \ll 1$, i.e. $l \gg 1$, we can obtain a good approximation to the solution of $*$ using the <u>WKBJ method</u>.

We try a solution of the form

$$\boxed{w = \exp(iS/\varepsilon)}$$

In that case $w' = (iS'/\varepsilon)\, e^{iS/\varepsilon}$ and

$$w'' = \left[iS''/\varepsilon - (S'/\varepsilon)^2\right] e^{iS/\varepsilon}$$

Thus

$$\left[i\varepsilon S'' - (S')^2 + \frac{a^2 - \mu^2 + \varepsilon^2}{(1-\mu^2)^2}\right] e^{iS/\varepsilon} = 0$$

$$\boxed{i\varepsilon S'' - (S')^2 + \frac{a^2 - \mu^2 + \varepsilon^2}{(1-\mu^2)^2} = 0} \quad *$$

Now let us write (* says we can) $\boxed{S = S_0 + \varepsilon S_1 + \cdots}$

This leads to, equating powers of ε:

$$-(S_0')^2 + \frac{a^2 - \mu^2}{(1-\mu^2)^2} = 0$$

$$2S_0' S_1' - iS_0'' = 0$$

Now look back at * for w.

Note: if $\mu^2 < a^2$, w is oscillatory
if $\mu^2 > a^2$, w is exponential

$\mu = \pm a$ are turning points for w.

For $l \gg 1$, we can always replace $l(l+1)$ by $(l+1/2)^2 = l(l+1) + 1/4$.

Let us define $\sin\theta_0 = \frac{|m|}{l+1/2}$ or $\theta_0 = \arcsin\left|\frac{m}{l+1/2}\right|$

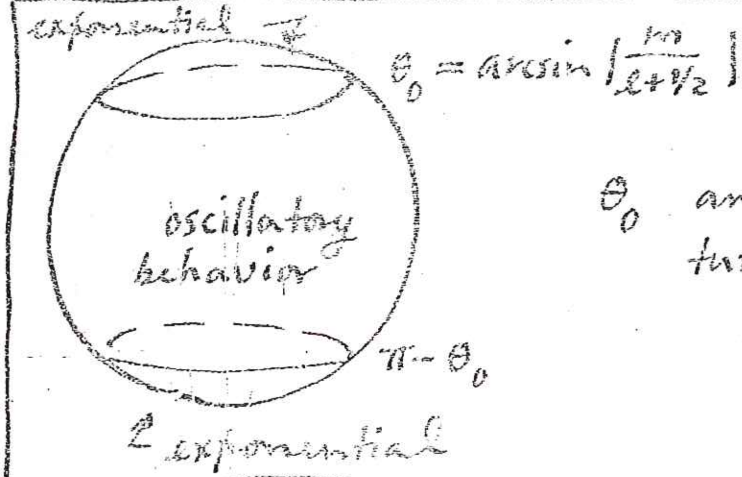
$$\boxed{\varepsilon^{-1} = \sqrt{l(l+1)} \approx (l+1/2)}$$

Then $\quad a \equiv \left[1 - \frac{m^2}{\ell(\ell+1)}\right]^{1/2} \approx \left[1 - \frac{m^2}{(\ell+1/2)^2}\right]^{1/2}$

i.e. $\quad a \cong \cos\theta_0$

The <u>turning points</u> are located at

$\mu^2 \approx a^2 \approx \cos^2\theta_0$, i.e at θ_0 and $\pi - \theta_0$.



θ_0 and $\pi - \theta_0$ are the turning colatitudes of $X_\ell^m(\theta)$

Between the turning colatitudes $\mu^2 < a^2$, X_ℓ^m is oscillatory, N of θ_0 and S of $\pi - \theta_0$, X_ℓ^m is exponential

In the <u>oscillatory</u> region from

$$S_0 = \pm \int^{\mu} \frac{\sqrt{a^2 - \mu^2}}{1 - \mu^2}\, d\mu \quad , \quad \text{i.e.}$$

The solution for S_1 gives $S_1 = \frac{i}{4} \ln \frac{a^2 - \mu^2}{(1-\mu^2)^2}$ apart from an additive constant.

This is because if $S_0' = Q^{1/2}$

$$S_0'' = \frac{1}{2} Q^{-1/2} Q'$$

$$S_1' = \frac{i}{2}\frac{S_0''}{S_0'} = \frac{i}{4}\frac{Q'}{Q} = \frac{i}{4}(\ln Q)'$$

We conclude that in the oscillatory region w is of the form

$$w = A\frac{(1-\mu^2)^{1/2}}{(a^2-\mu^2)^{1/4}} \exp\left[\pm\frac{i}{\varepsilon}\int^{\mu}\frac{(a^2-\mu^2)^{1/2}}{1-\mu^2}\,d\mu\right]$$

This is because $e^{iS/\varepsilon} = e^{iS_1}\, e^{iS_0/\varepsilon} \ldots$

$$= Q^{-1/4}\, e^{iS_0/\varepsilon}.$$

Since $y = (1-\mu^2)^{-1/2}\, w$, we find finally that:

$$X_\ell^m \sim A(a^2-\mu^2)^{-1/4}\exp\left[\pm\frac{i}{\varepsilon}\int^{\mu}\frac{(a^2-\mu^2)^{1/2}}{1-\mu^2}\,d\mu\right]$$

in the oscillatory region

here A is a <u>complex</u> constant and there is an unknown lower limit of integration.

We must select these to make the asymptotic formula agree with the known properties of X_ℓ^m.

Consider the two <u>symmetries</u>

$$\boxed{\begin{aligned} X_\ell^{-m}(\theta) &= (-1)^m\, X_\ell^m(\theta) \\ X_\ell^m(\pi-\theta) &= (-1)^{\ell+m}\, X_\ell^m(\theta) \end{aligned}}$$

These require that we write

$$X_\ell^m(\theta) \sim A\,(a^2-\mu^2)^{-1/4} \cos\left[\varepsilon^{-1}\int_0^\mu \frac{(a^2-\mu^2)^{1/2}}{1-\mu^2}\,d\mu \pm (\ell+m)\pi/2\right]$$

where A is real since X_ℓ^m is.

Check that these are then satisfied. The first is since

$$\begin{aligned} &\cos\left[\chi \pm (\ell-m)\pi/2\right] \\ &\quad= \cos\left[\chi \pm (\ell+m)\pi/2 \mp m\pi\right] \\ &\quad= (-1)^m \cos\left[\chi \pm (\ell+m)\pi/2\right]. \end{aligned}$$

Consider now the second of $\pm$.

$$X_\ell^m(\pi-\theta) = A\,(a^2-\mu^2)^{-1/4} \cos\Big[\varepsilon^{-1}\int_0^{-\mu}\frac{(a^2-\mu^2)^{1/2}}{1-\mu^2}\,d\mu$$

$$\pm (\ell+m)\pi/2\Big],$$

Change variable of integration to $-\mu$. Get

$$\cos[-\chi \pm (\ell+m)\pi/2]$$
$$= \cos[-(\chi \pm (\ell+m)\pi/2) \pm (\ell+m)\pi]$$
$$= (-1)^{\ell+m} \cos[\chi \pm (\ell+m)\pi/2].$$

Note that:

$\ell+m$ odd: $\quad X_\ell^m(\pi/2) = 0$

$\ell+m$ even: $\quad \operatorname{sgn} X_\ell^m(\pi/2) = (-1)^{\frac{\ell+m}{2}}.$

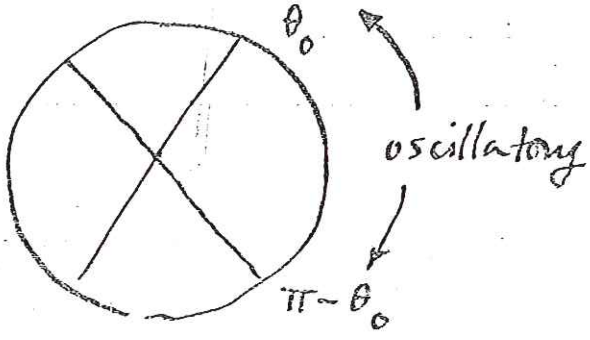
To find A, we make use of the <u>normalization</u>

$$\int_\Omega Y_\ell^m Y_\ell^{m*}\,dA = 1, \quad \text{i.e.}$$

$$2\pi\int_0^\pi (X_\ell^m(\theta))^2 \sin\theta\,d\theta = 1$$

The $\cos[\cdot]$ is very wiggly, so $\cos^2[\cdot] \sim 1/2$. Thus

We integrate only over the oscillatory portion. We assume it is exponentially small outside



Thus
$$2\pi \cdot \tfrac{1}{2} A^2 \int_{\theta_0}^{\pi-\theta_0} (a^2-\mu^2)^{-1/2} \sin\theta \, d\theta =$$

$$\text{let } x = a^{-1}\sin\theta = \frac{\sin\theta}{\sin\theta_0}$$

Then
$$\pi A^2 \int_{-1}^{1} (1-x^2)^{-1/2} \, dx = 1, \text{ or}$$

$$\boxed{A = \frac{1}{\pi}}$$

Thus, letting $\varepsilon^{-1} = \sqrt{\ell(\ell+1)} \sim (\ell + 1/2)$

$$X_\ell^m(\theta) \sim \frac{1}{\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4}$$

$$\cos\left[(\ell+1/2)\int_0^\mu \frac{(a^2-\mu^2)^{1/2}}{1-\mu^2}\, d\mu \pm (\ell+m)\frac{\pi}{2}\right]$$

We must still choose between $\pm \pi/2$.

The correct choice is — but why?

To see this requires consideration of the <u>connection formula</u>, since the one other thing we know, is how $X_\ell^m(\theta)$ starts out growing at $\theta = 0$ and $\theta = \pi$. We shall not go through this. With the correct choice, we obtain finally:

In the oscillatory regime $\theta_0 < \theta < \pi - \theta_0$ where $\theta_0 = \arcsin |m/(\ell + 1/2)|$

$$X_\ell^m(\theta) \sim \frac{1}{\pi} \left[1 - \frac{m^2}{(\ell+1/2)^2} - \cos^2\theta \right]^{-1/4}$$

$$\cos \left[(\ell + 1/2) \int_0^{\cos\theta} (1-\mu^2)^{-1} \left(1 - \frac{m^2}{(\ell+1/2)^2} - \mu^2\right)^{1/2} d\mu \right.$$

$$\left. - (\ell+m)\pi/2 \right]$$

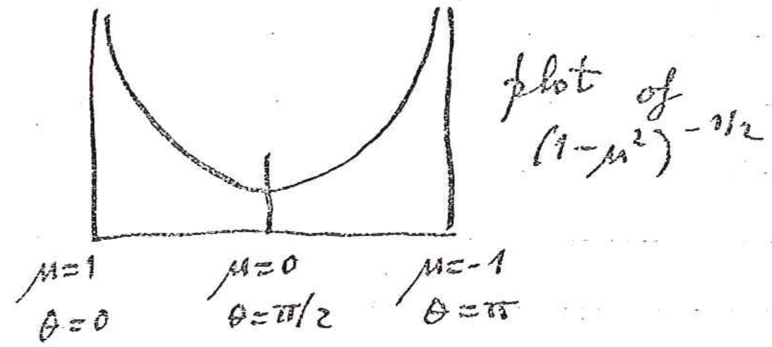
* in the limit $\ell \gg 1$

One other way to convince yourself the correct choice is — is to consider the case $m \ll \ell + 1/2$.

For $m \ll (l+1/2)$:

$$\left[1 - \frac{m^2}{(l+1/2)^2} - \cos^2\theta\right] \sim \left[1 - \cos^2\theta\right] = \sin^2\theta$$

$$\int_0^{\mu} (1-\mu^2)^{-1/2}\, d\mu = \pi/2 - \theta$$



$$\cos\left[(l+1/2)(\pi/2-\theta) - (l+m)\pi/2\right]$$
$$= \cos\left[(l+1/2)\theta + m\pi/2 - \pi/4\right]$$

Thus for $m \ll l$ and $\varepsilon \gg l^{-1}$, in the region $\varepsilon \leq \theta \leq \pi - \varepsilon$,

$$X_l^m(\theta) \sim \frac{1}{\pi}(\sin\theta)^{-1/2} \cos\left[(l+1/2)\theta + m\pi/2 - \pi/4\right]$$

We shall find this useful in the study of surface wave propagation.

In the integral make the change of variables

$$\mu = \cos\theta$$
$$d\mu = -\sin\theta \, d\theta$$

$$\int_{\pi/2}^{\theta} \sqrt{\sin^2\theta - \frac{m^2}{(l+1/2)^2}} \; \frac{-\sin\theta \, d\theta}{\sin^2\theta}$$

$$= -\int_{\pi/2}^{\theta} \sqrt{1 - \frac{m^2}{(l+1/2)^2 \sin^2\theta}} \; d\theta$$

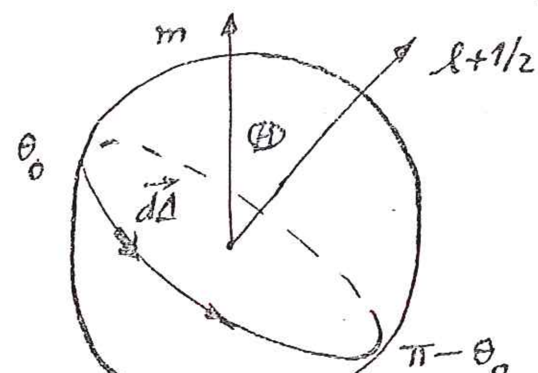
Thus can write * in the form

$$\chi_l^m(\theta) \sim \frac{1}{\pi} (\sin^2\theta - \sin^2\theta_0)^{-1/4}$$

$$\cos\left[\int_{\pi/2}^{\theta} p_\theta \, d\theta + (l+m)\frac{\pi}{2} \right]$$

where $p_\theta^2 = (l+1/2)^2 - m^2/\sin^2\theta$.

Consider a particle going around an orbit with angular momentum $l+1/2$ inclined at an angle Θ to the $\hat{z}$ axis, where



$$\cos\Theta = \frac{m}{l+1/2}.$$

The $\hat{\phi}$ component of angular momentum is

$$p_\phi = m/\sin\theta$$

At the turning colatitudes θ_0 and $\pi - \theta_0$,

$$p_\phi = \pm(l + 1/2), \quad \begin{array}{l} + \ \text{for } m > 0 \\ - \ \text{for } m < 0 \end{array}$$

$m > 0$: ccw from above

$m < 0$: cw from above.

The $\hat{\theta}$ component is $p_\theta = \pm\sqrt{(l+1/2)^2 - m^2/\sin^2\theta}$

Consider $\underline{p} \cdot \underline{d\Delta}$ along the path (see figure)

$$\underline{p} \cdot \underline{d\Delta} = p_\theta \, d\theta + p_\phi \sin\theta \, d\phi$$

$$= p_\theta \, d\theta + m \, d\phi = (l + 1/2) \, d\Delta$$

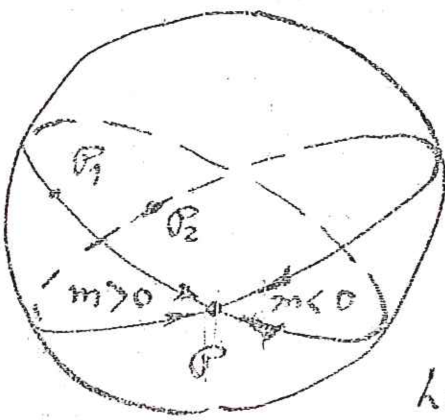
Consider $Y_l^m(\theta,\phi) \sim \frac{1}{\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4}$

$$\cos\left[\int_{\pi/2}^{\theta} p_\theta \, d\theta + (l+m)\pi/2\right] e^{im\phi}$$

Write $\cos\chi = \frac{1}{2}\left(e^{i\chi} + e^{-i\chi}\right)$

With a bit of manipulation this leads to another form for Y_l^m.

Given $P = (\theta, \phi)$ first construct the two unique great circles through P and tangent to the turning colatitudes. Pick any two points $P_1 = (\theta_1, \Phi_1)$ and $P_2 = (\theta_2, \Phi_2)$ one on each great circle. Then



$$Y_\ell^m \sim \frac{1}{2\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4} \sum_{1,2} e^{i[(\ell+1/2)\Delta + \Phi - N\pi/2]}$$

$m>0$: traverse in rt. hand direction $m<0$: in left hand direction.

where Δ is the distance from P_1 or P_2 to P and N is the number of intervening <u>caustic</u> encounters.

Homework: show the above.

The turning colatitudes are <u>caustics</u>: the envelopes of adjacent rays all with $\cos\Theta = m/(\ell+1/2)$

Recall that Y_ℓ^m expresses the angular dependence of the quantum mechanical wavefcn of a system with angular momentum quantum numbers $L^2 = l(l+1)$ and $L_z = m$.

We see that in the limit of large l, Y_l^m has all the properties of a classical wave with wave number (on Ω) $k = l + 1/2$ going around great circles inclined at $\cos\Theta = m/(l+1/2)$

This is an example of the <u>corresponce principle</u>.

$|Y_l^m|^2$ gives the q.m. <u>probability</u> of finding the orbiting particle at a point θ, ϕ

$|Y_l^m(\theta,\phi)|^2 = (X_l^m(\theta))^2$. This oscillates rapidly as a fcn of θ, but if we consider the <u>average</u> of these rapid oscillations, we find (factor of 2π from $\int_0^{2\pi} d\phi$

$$\boxed{2\pi\overline{(X_l^m(\theta))^2} = \frac{1}{\pi}(\sin^2\theta - \sin^2\theta_0)^{-1/2} = P(\cos\theta)}$$

This is exactly the <u>classical</u> probability of finding the orbiting particle at some value of $\cos\theta$ in its orbit.

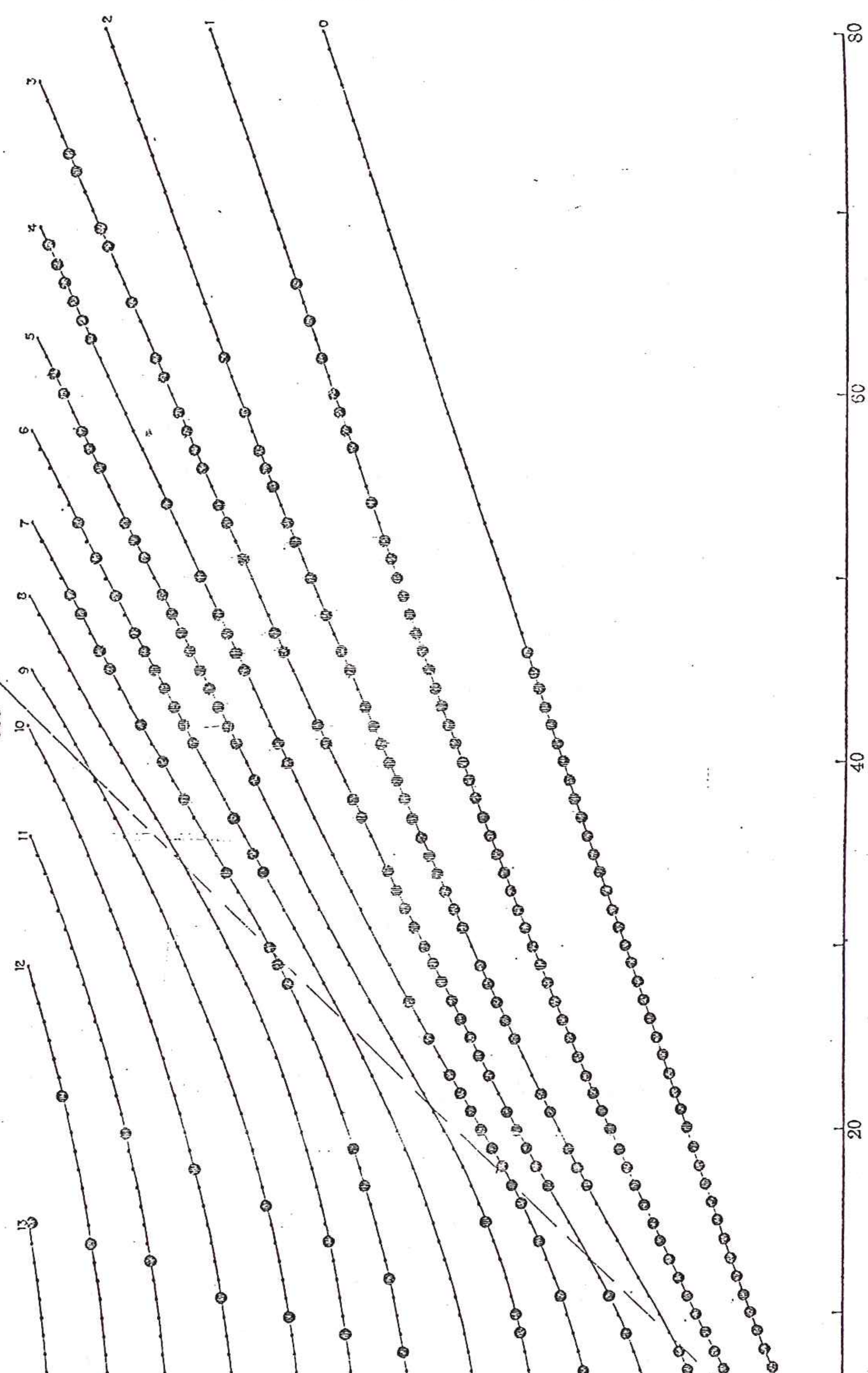
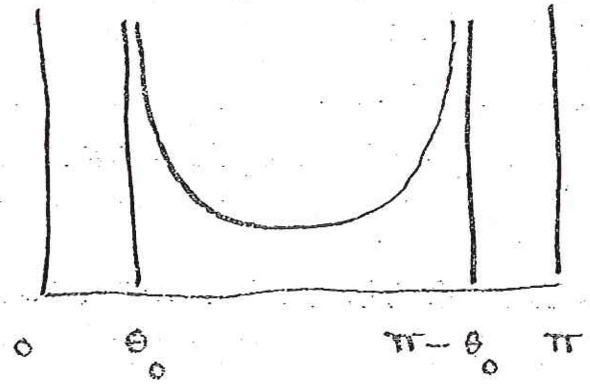


FIGURE 16. Toroidal normal modes in the (ω, l) plane. The large dots indicate observed modes used in the inversions. Most of the toroidal overtones identified by Brune & Gilbert (1964) fall outside the range of the figure. The dashed line designated 'ScS' divides the modes into two groups according to the normal mode–body wave analogy: modes to the left of this line correspond to $(ScS)_H$ reflexions, those to the right correspond to mantle S_H waves.

The classical probability vanishes N and S of the turning colatitudes.



$$2\pi\,\overline{\left(X_l^m(\theta)\right)^2} = \frac{1}{\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/2}$$

The classical probability density in $\cos\theta$ of a particle is

$$P(\cos\theta) \;\propto\; \left| d(\cos\theta(t))/dt \right|^{-1}$$

$$(d\cos\theta/dt)^{-1} = \left| -\sin\theta \;\, d\theta/dt \right|^{-1}$$

$$\propto \left| -\sin\theta \; p_\theta \right|^{-1} \;\propto\; (\sin^2\theta - \sin^2\theta_0)^{-1/2}$$

Finally, we wish to derive a form suitable for numerical calculations.

We go back to * on page 216.
With $a^2 \equiv 1 - m^2/(l+1/2)^2$

$$X_l^m(\theta) \sim \frac{1}{\pi}(\sin^2\theta - \sin^2\theta_0)^{-1/4}$$

$$\cos\Big[(l+1/2) \int_0^\mu (1-\mu^2)^{-1}(1-a^2-\mu^2)^{1/2}\, d\mu - (l+m)\pi/2 \Big]$$

Write integrand as

$$\frac{(1-a^2-\mu^2)^{1/2}}{1-\mu^2} = \frac{1}{(1-a^2-\mu^2)^{1/2}} - \frac{a^2}{(1-\mu^2)(1-a^2-\mu^2)^{1/2}}$$

$$\int_0^\mu \frac{1}{(1-a^2-\mu^2)^{1/2}}\, d\mu = \arcsin \frac{\mu}{\sqrt{1-a^2}}$$

$$= \arcsin(\cos\theta/\cos\theta_0)$$

Now consider $-a^2 \int_0^\mu (1-\mu^2)^{-1}(1-a^2-\mu^2)^{-1/2}\, d\mu$

Try the inspired substitution

$$\boxed{\mu(1-a^2-\mu^2)^{-1/2} = t}\ *$$

$$1-\mu^2 = \frac{1+a^2t^2}{1+t^2}$$

and differentiating

$$d\mu\,(1-a^2-\mu^2)^{-1/2} + \mu^2\,d\mu\,(1-a^2-\mu^2)^{-3/2} = dt$$

$$d\mu\,(1-a^2) = dt\,(1-\mu^2-M^2)^{3/2}$$

$$= dt\left(\frac{1+M^2t^2-M^2-M^2t^2}{1+t^2}\right)^{3/2}$$

$$d\mu = dt\,(1+t^2)^{-3/2}\,\sqrt{1-a^2}$$

So

$$\frac{d\mu}{(1-\mu^2)\sqrt{1-a^2-\mu^2}} = \frac{dt\sqrt{1-a^2}}{(1+t^2)^{3/2}}\cdot\frac{1+t^2}{1+a^2t^2}\cdot\sqrt{\frac{1+t^2}{1-a^2}}$$

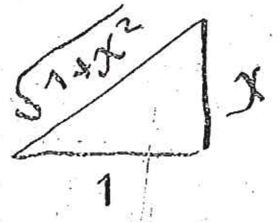
$$= \frac{dt}{1+a^2t^2}$$

Thus get $\displaystyle\int_0^{\mu/\sqrt{1-a^2-\mu^2}} dt\,/\,(1+a^2t^2)$

$$= \frac{1}{a}\arctan\left(\frac{a\mu}{\sqrt{1-a^2-\mu^2}}\right)$$

$$= \frac{1}{a}\arctan\left(\frac{a\cos\theta}{\sqrt{\cdots}}\right)$$

Now $\arctan X = \arcsin\left(\dfrac{X}{\sqrt{1+X^2}}\right)$, so



$$-a \arctan\left(\frac{a\cos\theta}{\sqrt{\cos^2\theta_0 - \cos^2\theta}}\right)$$

$$= -a \arcsin \frac{a\cos\theta}{\sin\theta\cos\theta_0}$$

The integrand is thus just

$$(l+1/2)\left[\underline{\arcsin(\cos\theta/\cos\theta_0)} - \underline{\underline{\frac{m}{l+1/2}}}\right.$$
$$\left.\underline{\underline{\arcsin\left(\frac{a\cos\theta}{\sin\theta\cos\theta_0}\right)}}\right] - \underline{(l+\underline{\underline{m}})\frac{\pi}{2}}$$

The underlined terms combine to give

$$-(l+1/2)\arccos(\cos\theta/\cos\theta_0) + \pi/4$$

Consider $m>0$, so that $a = \sin\theta_0$.
Then the underlined terms give

$$-m \arccos(\sin\theta_0\cos\theta/\sin\theta\cos\theta_0) - m\pi$$

Now use $\cos[X] = \cos[-X]$

Summary: in the oscillatory region $\theta_0 < \theta < \pi - \theta_0$, in the limit of large l, for $m > 0$,

$$X_l^m(\theta) \sim \frac{1}{\pi}\left(\sin^2\theta - \sin^2\theta_0\right)^{-1/4}$$

$$\cos\Big[(l+1/2)\arccos\left(\cos\theta/\cos\theta_0\right)$$

$$- m\arccos\left(\sin\theta_0\cos\theta/\sin\theta\cos\theta_0\right)$$

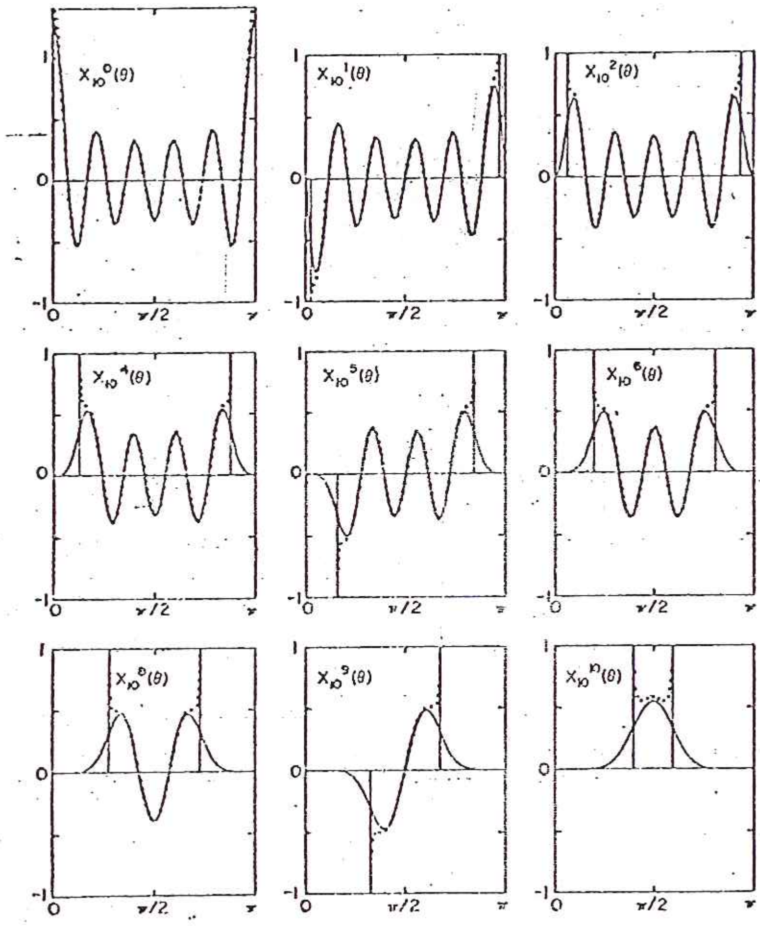
$$+ m\pi - \pi/4\Big]$$

where $\sin\theta_0 = m/(l+1/2)$

Can find $X_l^{-m}(\theta)$ from $X_l^{-m} = (-1)^m X_l^m$.

$\star$ is very easy to compute.

The agreement with the exact $X_l^m(\theta)$ is very good for $l = 10$.



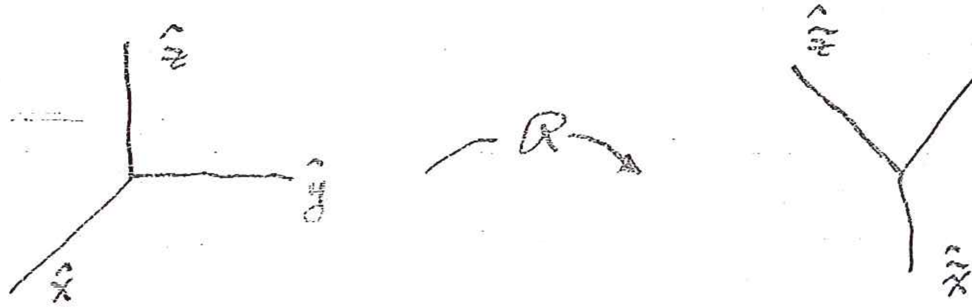
$X_{10}^{0}(\theta)$
$X_{10}^{1}(\theta)$
$X_{10}^{2}(\theta)$
$X_{10}^{3}(\theta)$
$X_{10}^{4}(\theta)$
$X_{10}^{5}(\theta)$
$X_{10}^{6}(\theta)$
$X_{10}^{7}(\theta)$
$X_{10}^{8}(\theta)$
$X_{10}^{9}(\theta)$
$X_{10}^{10}(\theta)$
1
0
-1
0
$\pi/2$
π

Figure 18

27. <u>Change of basis in $\mathcal{H}_\ell(\Omega)$ induced by a change in basis of $\mathbb{R}^3$.</u>

Every CAS $\hat{x}, \hat{y}, \hat{z}$ in $\mathbb{R}^3$ generates a canonical basis $\{Y_\ell^m(\theta,\phi), -\ell \le m \le \ell\}$ of $\mathcal{H}_\ell(\Omega)$.

Consider two CAS $\hat{x}, \hat{y}, \hat{z}$ and $\hat{\tilde{x}}, \hat{\tilde{y}}, \hat{\tilde{z}}$. Let $\mathcal{R}$ denote the rotation which takes one into the other.



Let $Y_\ell^m(\hat{r})$ and $\tilde{Y}_\ell^{\tilde{m}}(\hat{r})$ be the corresponding surface spherical harmonics on Ω.

Both are an o.n. basis of $\mathcal{H}_\ell(\Omega)$. Therefore $\exists$ a $2\ell+1 \times 2\ell+1$ unitary matrix $D_{\tilde{m}m}^{(\ell)}(\mathcal{R})$ such that

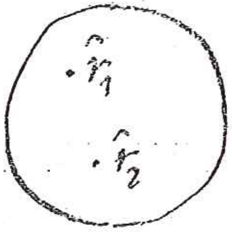
$$\boxed{Y_\ell^m = \sum_{\tilde{m}=-\ell}^{\ell} D_{\tilde{m}m}^{(\ell)} \tilde{Y}_\ell^{\tilde{m}}}$$

The matrix is <u>unitary</u>, i.e.

$$\boxed{\sum_{\tilde{m}=-\ell}^{\ell} \mathcal{D}^{(\ell)}_{\tilde{m}m} \, \mathcal{D}^{(\ell)\,*}_{\tilde{m}m'} = \delta_{mm'}}$$

Expressions for calculating the matrix elements $\mathcal{D}^{(\ell)}_{\tilde{m}m}$ in terms of the <u>Euler angles</u> defining $\mathcal{R}$ are given, e.g. by Edmonds.

Consider this basis transformation at two arbitrary points $\hat{r}_1$ and $\hat{r}_2 \in \Omega$.



$$Y_\ell^m(\hat{r}_1) = \sum_{\tilde{m}=-\ell}^{\ell} \mathcal{D}^{(\ell)}_{\tilde{m}m} \, \tilde{Y}_\ell^{\tilde{m}}(\hat{r}_1)$$

$$Y_\ell^m(\hat{r}_2) = \sum_{\tilde{m}=-\ell}^{\ell} \mathcal{D}^{(\ell)}_{\tilde{m}m} \, \tilde{Y}_\ell^{\tilde{m}}(\hat{r}_2)$$

Let us denote:

$$Y(\hat{r}) \equiv \begin{bmatrix} Y_\ell^{-\ell}(\hat{r}) \\ \vdots \\ Y_\ell^{\ell}(\hat{r}) \end{bmatrix}$$

and employ matrix notation.

Then $\quad Y(\hat{r}_1) = \tilde{Y}(\hat{r}_1)\,\mathcal{D}$

$\qquad\quad Y(\hat{r}_2) = \tilde{Y}(\hat{r}_2)\,\mathcal{D}$

Note the "perversity of the indices".

Now consider: $\quad Y(\hat{r}_1)\,Y^{\dagger}(\hat{r}_2)$

If $\quad Y = [\;]$, then $Y^{\dagger} = [\quad]^{*}$

$$Y(\hat{r}_1)\,Y^{\dagger}(\hat{r}_2) = \left(\tilde{Y}(\hat{r}_1)\,\mathcal{D}\right)\left(\tilde{Y}(\hat{r}_2)\,\mathcal{D}\right)^{\dagger}$$

$$= \tilde{Y}(\hat{r}_1)\,\mathcal{D}\mathcal{D}^{\dagger}\,\tilde{Y}^{\dagger}(\hat{r}_2)$$

$$= \tilde{Y}(\hat{r}_1)\,\tilde{Y}^{\dagger}(\hat{r}_2) \quad , \quad \text{since } \mathcal{D}\mathcal{D}^{\dagger} = I.$$

<u>unitary</u>: inverse is adjoint.

Thus

$$\sum_{m=-l}^{l} Y_l^{m}(\hat{r}_1)\,Y_l^{m\,*}(\hat{r}_2) = \sum_{\tilde{m}=-l}^{l} \tilde{Y}_l^{\tilde{m}}(\hat{r}_1)\,\tilde{Y}_l^{\tilde{m}\,*}(\hat{r}_2)$$

true for any $\hat{r}_1, \hat{r}_2$

This is merely the statement that the change of basis in $\mathcal{H}_\ell(\Omega)$ does not change the lengths or geometrical relationships between any vectors, i.e. it is a rotation, i.e. $*$ is the counterpart of

$$\hat{r}_1 = (x_1, y_1, z_1) = (\tilde{x}_1, \tilde{y}_1, \tilde{z}_1)$$
$$\hat{r}_2 = (x_2, y_2, z_2) = (\tilde{x}_2, \tilde{y}_2, \tilde{z}_2)$$

Then $\hat{r}_1 \cdot \hat{r}_2 = (x_1 x_2 + y_1 y_2 + z_1 z_2)$
$= (\tilde{x}_1 \tilde{x}_2 + \tilde{y}_1 \tilde{y}_2 + \tilde{z}_1 \tilde{z}_2)$

The sum $*$ is independent of the basis elements used.

Now take $\hat{\tilde{x}}, \hat{\tilde{y}}, \hat{\tilde{z}}$ s.t. $\hat{\tilde{z}} \equiv \hat{r}_1$.
Then

$$\tilde{Y}_\ell^{\tilde{m}}(\hat{r}_1) = \tilde{Y}_\ell^{\tilde{m}}(\hat{\tilde{z}}) = \begin{cases} 0 & \text{if } \tilde{m} \neq 0 \\ \sqrt{\frac{2\ell+1}{4\pi}} & \text{if } \tilde{m} = 0 \end{cases}$$

Thus

$$\sqrt{\frac{2\ell+1}{4\pi}}\, \tilde{Y}_\ell^{0*}(\hat{r}_2) = \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{r}_1)\, Y_\ell^{m*}(\hat{r}_2)$$

but $\tilde{Y}_\ell^{0*}(\hat{r}_2) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\Theta)$, $\cos\Theta = \hat{\tilde{z}} \cdot \hat{r}_2$
$= \hat{r}_1 \cdot \hat{r}_2$.

Thus, the spherical harmonic addition theorem

$$\sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{r}_1)\, Y_\ell^{m*}(\hat{r}_2) = \frac{2\ell+1}{4\pi} P_\ell(\hat{r}_1 \cdot \hat{r}_2)$$

for any $\hat{x}, \hat{y}, \hat{z}$. $\ast$

This a very useful result.
As an aside, can bound $Y_\ell^m(\hat{r})$.

Choose $\hat{r}_1 = \hat{r}_2$ in $\ast$. Then, calling $\hat{r}_1 = \hat{r}_2 = \hat{r}$

$$\sum_{m=-\ell}^{\ell} |Y_\ell^m(\hat{r})|^2 = \frac{2\ell+1}{4\pi}$$

Thus every $Y_\ell^m(\hat{r})$ must satisfy

$$|Y_\ell^m(\hat{r})| \leq \sqrt{\frac{2\ell+1}{4\pi}}$$

The bound is achieved by Y_ℓ^0 at $\hat{r} = \hat{z}$, but nowhere else.

28. Expansions of arbitrary functions in series of spherical harmonics

We now examine to what extent an "arbitrary" function $f : \Omega \to \mathbb{C}$ can be written as a sum of surface spherical harmonics.

Already know one result (followed from Weierstrass theorem).

If $f: \Omega \to \mathbb{C}$ is <u>continuous</u>, then $\exists$ $Y_\ell(\hat{r}) \in \mathcal{H}_\ell(\Omega)$ s.t.

$$\left| f(\hat{r}) - \sum_{\ell=0}^{L} Y_\ell(\hat{r}) \right| < \varepsilon ,$$

i.e. a cont. fcn $f: \Omega \to \mathbb{C}$ can be <u>uniformly</u> approximated. This theorem merely asserts existence. How could we <u>determine</u> the $Y_\ell(\hat{r})$?

<u>Formally</u>, say

$$f(\hat{r}) = \sum_{\ell=0}^{\infty} Y_\ell(\hat{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\hat{r})$$

Each $Y_\ell(\hat{r})$ is a lin. comb. of the canonical basis elements $Y_\ell^m(\hat{r})$. Can use <u>orthogonality</u> to find the coeff. f_ℓ^m.

Take inner product with an arbitrary $Y_{\ell'}^{m'}(\hat{r})$

$$(f, Y_{l'}^{m'}) = \left(\sum_l \sum_m f_l^m Y_l^m , Y_{l'}^{m'}\right)$$

$$= \sum_l \sum_m f_l^m (Y_l^m, Y_{l'}^{m'}) =$$

$$\sum_l \sum_m f_l^m \delta_{ll'} \delta_{mm'} = f_{l'}^{m'}$$

Define $\quad f_l^m \equiv (f, Y_l^m) = \int_\Omega f(\hat{r}) Y_l^{m*}(\hat{r})\, dA$

These will be called the <u>Laplace coefficients</u>

Also define L^{th} <u>partial sum</u>

$$f_L(\hat{r}) = \sum_{l=0}^{L} \sum_{m=-l}^{l} f_l^m Y_l^m(\hat{r})$$

Our theorem asserts the existence of a fcn of the <u>form</u> $f_L(\hat{r})$ s.t.

$$\lim_{L\to\infty} f_L(\hat{r}) = f(\hat{r}) \quad \text{uniformly on } \Omega$$

if $f(\hat{r})$ is <u>continuous</u>.

Life would be nice if the fcns $f_L(\hat{r})$ defined above were exactly the one and only such fcns whose existence is asserted. Mais, malheureusement, ce n'est pas vrai

Can be shown that $\exists$ continuous fns $f(\hat{r})$ for which $\lim_{L\to\infty} f_L(\hat{r})$ fails to exist at an ∞ no. of pts.

Two questions now arise.

1. What stronger cond. on f is sufficient to insure convergence of $f_L(\hat{r})$ to $f(\hat{r})$ as $L \to \infty$?
   Answer: if $f: \Omega \to \mathbb{C}$ is continuously differentiable (i.e. if $\nabla_1 f$ is cont.) then

$$\lim_{L\to\infty} f_L(\hat{r}) = f(\hat{r}).$$

   Somewhat weaker but more complicated conds. will also suffice.

2. f cont. does not insure pointwise convergence, but we know $\exists$ a series which converges uniformly. Can such a series be exhibited? Yes. If one defines the so-called Césaro mean as the arithmetic mean of partial sums of Laplace coefficients i.e.

$$\sigma_L(\hat{r}) = \frac{1}{L+1}\left[f_0(\hat{r}) + f_1(\hat{r}) + \cdots f_L(\hat{r})\right],$$

then it can be shown that

$\sigma_L(\hat{r}) \to f(\hat{r})$ uniformly at all $\hat{r}$ where f is continuous.

So much for pointwise convergence. Actually, the Laplace series is much more naturally associated with another type of convergence (convergence in the mean)

Consider the problem of making a least squares approximation to a fcn f by a finite l.c. of Y_ℓ^m's.

Take $f \in \mathcal{L}_2(\Omega)$, any square int. fcn. As long as $f \in \mathcal{L}_2(\Omega)$, the Laplace coefficients

$$f_\ell^m \equiv (f, Y_\ell^m) \quad \text{exist.}$$

We wish to determine coefficients g_ℓ^m s.t.

$$\left\| f - \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} g_\ell^m Y_\ell^m \right\|^2 = \int_\Omega \left| f - \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} g_\ell^m Y_\ell^m \right|^2 dA$$

is a <u>minimum</u>.

But $\int_\Omega |f - \sum\sum g_\ell^m Y_\ell^m|^2 \, dA$

$$= \int_\Omega (f - \sum\sum g_\ell^m Y_\ell^m)(f - \sum\sum g_\ell^m Y_\ell^m)^* \, dA$$

$$= \int_\Omega |f|^2 dA + \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} \left[-g_\ell^{m*} f_\ell^m - g_\ell^m f_\ell^{m*} + |g_\ell^m|^2 \right]$$

$$= \int_\Omega |f|^2 dA + \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} |f_\ell^m - g_\ell^m|^2 - \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} |f_\ell^m|^2$$

The minimum is attained when $g_\ell^m = f_\ell^m$. This is the <u>best approximation</u> by <u>a finite no. of terms</u>.

The r.m.s. error in making this best approximation is

$$\| f - f_L \|^2 = \int_\Omega | f(\hat{r}) - f_L(\hat{r}) |^2 \, dA$$

where $f_L(\hat{r}) = \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\hat{r})$.

$$= \|f\|^2 - \sum_{l=0}^{L} \sum_{m=-l}^{l} |f_l^m|^2$$

i.e. the length of f squared − the sum of the squared lengths of all the orthogonal projections of f onto the o.n. basis elements Y_l^m.

<u>Theorem</u> : it can be shown that for all $f \in \mathcal{L}_2(\Omega)$, this error can be made as small as one desires, i.e.

$$\lim_{L \to \infty} \|f - f_L\|^2 = 0.$$

The content of this thm. is often stated by saying that every $f \in \mathcal{L}_2(\Omega)$ can be written in the form

$$\boxed{f(\hat{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\hat{r}) \qquad *}$$

in a <u>unique manner</u>, and the series is said to <u>converge in the mean</u> (this is the sense in which the = sign in * is interpreted). The set of all $Y_l^m(\hat{r})$ is said to be an o.n. basis for the <u>Hilbert space</u> $\mathcal{L}_2(\Omega)$.

29. The representation of tangent vectors

We have considered the expansion of a *scalar* function $f : \Omega \to C$ in terms of s.s.h. $Y_\ell^m(\hat{r})$.

Main result: if $f \in \mathcal{L}_2(\Omega)$, then

$$* \quad f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m \qquad \text{where}$$

$$f_\ell^m = (f, Y_\ell^m) = \int_\Omega f Y_\ell^{m\,*}\, dA$$

and where $*$ is to be interpreted in the mean.

We consider now a vector-valued function $\underline{f}(\hat{\underline{r}})$:
$$\underline{f} : \Omega \to \underline{C} \equiv C^3 .$$

We would like to represent $\underline{f}$ in terms of three scalar fields, which we may then in turn represent as s.s.h. expansions. One possibility is obvious:

$$\underline{f} = f_r \hat{r} + f_\theta \hat{\theta} + f_\phi \hat{\phi} .$$

This is not however a very convenient representation.

It is useful to separate $\underline{f}$ into its radial and tangential parts.

$$\boxed{\begin{aligned}\underline{f}(\hat{r}) &= f_r(\hat{r})\,\hat{r} + \underline{t}(\hat{r})\\ \hat{r}\cdot\underline{t}(\hat{r}) &= 0\end{aligned}}$$

The problem now is to represent the tangent vector $\underline{t}(\hat{r})$ as a sum of two scalar fields.

Tangent vector representation theorem: If $\underline{t}:\Omega\to\mathbb{C}^3$ is continuously differentiable tangent vector field on Ω, then $\exists$ unique scalar fields $V(\hat{r})$ and $W(\hat{r})$ s.t.

$$\underline{t}(\hat{r}) = \underline{\nabla}_1 V(\hat{r}) - \hat{r}\times\underline{\nabla}_1 W(\hat{r})$$

and

$$\int_\Omega V(\hat{r})\,dA = \int_\Omega W(\hat{r})\,dA = 0$$

Notes: at every $\hat{r}$, $\nabla_1 V$ and $-\hat{r}\times\nabla_1 W$ are tangent vectors.

The $-$ sign is conventional (to agree with an older convention, established prior to this notation.)

Since $\underline{\Lambda} \equiv \hat{r} \times \underline{\nabla} = \hat{r} \times \underline{\nabla}_1$, can also write

$$\boxed{\underline{t} = \underline{\nabla}_1 V - \underline{\Lambda} W \qquad \star}$$

$$\boxed{\begin{aligned} \underline{\nabla}_1 &= \hat{\theta}\,\partial_\theta + (\sin\theta)^{-1}\hat{\phi}\,\partial_\phi \\ \underline{\Lambda} &= -(\sin\theta)^{-1}\hat{\theta}\,\partial_\phi + \hat{\phi}\,\partial_\theta \end{aligned}}$$

Proof: If $\star$ is true, then V and W must satisfy (taking $\underline{\nabla}_1 \cdot \star$ and $\underline{\Lambda} \cdot \star$, and using $\underline{\nabla}_1 \cdot \underline{\Lambda} = \underline{\Lambda} \cdot \underline{\nabla}_1 = 0$)

$$\begin{aligned} \nabla_1^2 V &= \underline{\nabla}_1 \cdot \underline{t} \\ -\Lambda^2 W &= \underline{\Lambda} \cdot \underline{t} \end{aligned}$$

But $\nabla_1^2 = \Lambda^2 = \dfrac{1}{\sin\theta}\,\partial_\theta (\sin\theta)\,\partial_\theta + \dfrac{1}{\sin^2\theta}\,\partial_\phi^2$

Thus

$$\boxed{\begin{aligned} \nabla_1^2 V &= \Lambda^2 V = \underline{\nabla}_1 \cdot \underline{t} \\ \nabla_1^2 W &= \Lambda^2 W = -\underline{\Lambda} \cdot \underline{t} \end{aligned}}$$

Formally we find

$$\boxed{\begin{aligned} V &= \Lambda^{-2}\,\underline{\nabla}_1 \cdot \underline{t} \\ W &= -\Lambda^{-2}\,\underline{\Lambda} \cdot \underline{t} \end{aligned}}$$

← write using ∇_1^{-2}

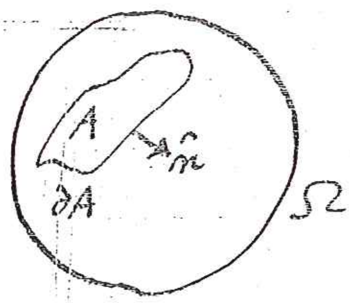
We are thus led to examine the <u>invertibility</u> of Λ^2. Consider

$$\nabla_1^2 f \quad \cancel{\Lambda^2} f = g. \qquad (*) \qquad\qquad \nabla_1^2 \psi = \chi \text{ in DFT}$$

If this is to be soluble for f, we must have

$$\int_\Omega g \, dA = 0 , \quad \text{since}$$

$$\int_\Omega \Lambda^2 f \, dA = \int_\Omega \nabla_1 \cdot \nabla_1 f \, dA = 0$$ <u>by Gauss' theorem on Ω</u>



> Gauss theorem on Ω states that if $\underline{t}$ is cont. diff. in A, then
>
> $$\int_A \nabla_1 \cdot \underline{t} \, dA = \int_{\partial A} \hat{n} \cdot \underline{t} \, dA$$

Thus a <u>necessary condition</u> for $(*)$ to have a soln is that

$$\int_\Omega g \, dA = 0$$

If $g(\vec{r}) \in \mathcal{L}_2(\Omega)$, we can write

$$\boxed{g = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} g_\ell^m Y_\ell^m . \quad ** \qquad \{\text{note} : g_0^0 = 0}$$

But since $\Lambda^2 Y_\ell^m = -\ell(\ell+1) Y_\ell^m$, the <u>unique</u> solution to $*$ is

$$\boxed{f = -\sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{g_\ell^m}{\ell(\ell+1)} Y_\ell^m \qquad ***}$$

This series is convergent in the mean <u>square</u> if that for g is, and this <u>uniquely</u> defines f.

To summarize: an operator Λ^{-2} can be defined on the space of all $\mathcal{L}_2(\Omega)$ fcns which average to zero on Ω, i.e. all $g \in \mathcal{L}_2(\Omega)$ s.t. $\int_\Omega g \, dA = 0$. If g is given by $**$ above, then $\Lambda^{-2} g$ is defined to be the f of $***$. The operator Λ^{-2} is <u>linear</u> and we have

$$\Lambda^{-2}\Lambda^2 g = \Lambda^2 \Lambda^{-2} g = g$$

Backus (1958) has given an <u>alternative</u> representation of $f = \nabla^{-2} g$, namely

$$\boxed{f(\hat{r}) = \frac{1}{4\pi} \int_\Omega g(\hat{r}') \ln (1 - \hat{r} \cdot \hat{r}') \, dA'}$$

Let us apply the above method of solving $\nabla_1^2 f = g$ or $\Lambda^2 f = g$ to

$$\nabla_1^2 V = \Lambda^2 V = \underline{\nabla}_1 \cdot \underline{t}$$
$$\nabla_1^2 W = \Lambda^2 W = -\underline{\Lambda} \cdot t$$

Both r.h.s. satisfy $\int_\Omega \underline{\nabla}_1 \cdot \underline{t}\, dA = -\int_\Omega \underline{\Lambda} \cdot \underline{t}\, dA = 0$,

i.e. both average to zero on Ω. Thus both have unique solutions, i.e. $\exists$ unique V, W defined by

$$V = \Lambda^{-2} \underline{\nabla}_1 \cdot \underline{t} \qquad *$$
$$W = -\Lambda^{-2} \underline{\Lambda} \cdot \underline{t}$$

and both V, W average to zero on Ω, i.e. $\int_\Omega V\, dA = \int_\Omega W\, dA = 0$.

We have thus shown that if V, W satisfying $\underline{t} = \underline{\nabla}_1 V - \underline{\Lambda} W$ and $\int_\Omega V dA = \int_\Omega W dA = 0$ exist, then they are unique, in fact they are given uniquely by * above. It remains to show that $\underline{t}$ is given in terms of V and W by

$$\underline{t} = \underline{\nabla}_1 V - \underline{\Lambda} W.$$

This question is settled by the following

$$\boxed{\begin{array}{l}\underline{\text{Lemma}}: \text{ Let } \underline{t} \text{ be a tangent vector field cont. differ. and bounded everywhere on } \Omega. \text{ Then} \\ \nabla_1 \cdot \underline{t} = \Lambda \cdot \underline{t} = 0 \text{ implies that } \underline{t} \equiv \underline{0}.\end{array}}$$

Proof: (Backus 1958)

Introduce the change of variable

$$\xi = -\ln(\csc\theta + \cot\theta)$$

Then $\partial_\xi = \sin\theta \, \partial_\theta$ *

The mapping $(\theta, \phi) \to (\xi, \phi)$ is the Mercator projection of Ω into the plane. To check *

$$\frac{d\xi}{d\theta} = -\frac{1}{\csc\theta + \cot\theta}\left(-\csc\theta\cot\theta - \csc^2\theta\right)$$

$$= \csc\theta \, , \quad \text{so} \quad \frac{d\theta}{d\xi} = \sin\theta$$

$$\text{i.e.} \quad \frac{\partial}{\partial\xi} = \sin\theta \frac{\partial}{\partial\theta}$$

Thus $\nabla_1 \cdot \underline{t} \equiv \frac{1}{\sin\theta} \partial_\theta (\sin\theta \, t_\theta)$

$$+ \frac{1}{\sin\theta} \partial_\phi t_\phi = 0$$

implies that

$$\sin\theta\ \partial_\theta(\sin\theta\, t_\theta) + \partial_\phi(\sin\theta\, t_\phi) = 0$$

i.e. $$\boxed{\partial_\xi(\sin\theta\, t_\theta) + \partial_\phi(\sin\theta\, t_\phi) = 0}$$

Likewise $\underline{\Lambda}\cdot\underline{t} = \underline{r}\cdot\nabla\times\underline{t}$

$$= \frac{1}{\sin\theta}\partial_\theta(\sin\theta\ t_\phi) - \frac{1}{\sin\theta}\partial_\phi t_\theta \quad \text{implies}$$

$\underline{\Lambda} = -\hat{r}\times\underline{\nabla}$, so $\underline{\Lambda}\cdot = \underline{r}\cdot\nabla\times$

$$\boxed{\partial_\xi(\sin\theta\, t_\phi) - \partial_\phi(\sin\theta\, t_\theta) = 0}$$

But these are precisely the Cauchy-Riemann equations in the plane $z = \xi + i\phi$ for the function $f(z) = \sin\theta\ (t_\theta - i t_\phi)$.

Thus $\sin\theta\ (t_\theta - i t_\phi)$ must be an entire function, i.e. regular everywhere in the z-plane except at ∞. Note it is periodic with period 2π in ϕ.

Since $\boxed{f(z) = \sin\theta\,(t_\theta - i t_\phi)}$ is

both <u>bounded</u> and entire, it must be a constant by <u>Liouville's</u> <u>theorem</u>. Since at

$$\zeta \to \infty \quad , \quad \text{i.e.} \quad \theta \to 0$$

$f(z)$ vanishes (because of the factor of $\sin\theta$), the constant must be zero, i.e.

$$f(z) = 0 \quad \text{for all } z.$$

Thus $t_\theta = t_\phi = 0$, or

$$\boxed{\underline{t} = \underline{0}}$$

<u>Cauchy</u> - <u>Riemann</u> equations: if $f(z) = u(x,y) + iv(x,y)$, then

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ at every z where $f(z)$ is regular

Now we can use this to show that

$$\underline{t} = \underline{\nabla}_1 V - \underline{\Lambda} W$$

Consider $\underline{t} - \nabla_1 V + \underline{\Lambda} W$.

$$\nabla_1 \cdot (\underline{t} - \nabla_1 V + \underline{\Lambda} W) = \nabla_1 \cdot \underline{t} - \nabla_1^2 V = 0$$

if $V \equiv \Lambda^{-2} \nabla_1 \cdot \underline{t}$

$$\underline{\Lambda} \cdot (t - \nabla_1 V + \underline{\Lambda} W) = \underline{\Lambda} \cdot t + \Lambda^2 W = 0$$

if $W \equiv \Lambda^{-2} (-\underline{\Lambda} \cdot \underline{t}$

Thus, by the lemma, $t - \nabla_1 V + \underline{\Lambda} W = 0$

or

$$\boxed{\underline{t} = \underline{\nabla}_1 V - \underline{\Lambda} W}$$

This completes the proof of the tangent vector representation theorem.

30. Vector spherical harmonics

Consider a vector field $\underline{s}$ on Ω. We now know we can represent it in terms of three scalar fields as follows

$$\boxed{\underline{s} = \hat{r}U + \nabla_1 V - \hat{r} \times \nabla_1 W}$$

where $U(\hat{r})$ is what we previously called s_r.

Now all three of the scalar fields $U(\hat{r})$, $V(\hat{r})$ and $W(\hat{r})$ can be expanded in s.s.h.

$$U(\hat{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_l^m Y_l^m(\hat{r})$$

Recall that to insure uniqueness we must specify $\int_\Omega V(\hat{r})\,dA = \int_\Omega W(\hat{r})\,dA = 0$

Thus

$$V(\hat{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} V_l^m Y_l^m(\hat{r})$$

$$W(\hat{r}) = \sum^{\infty} \sum^{l} W_l^m Y_l^m(\hat{r})$$

Note lower limit $l = 1$.
We are thus led to, with $V_0^0 = W_0^0 = 0$

$$\underline{s}(\hat{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[U_l^m \, \hat{r} Y_l^m(\hat{r}) + V_l^m \nabla_1 Y_l^m(\hat{r}) + W_l^m \left(-\hat{r} \times \nabla_1 Y_l^m(\hat{r}) \right) \right]$$

It is conventional to define

$$\boxed{\begin{aligned} \underline{P}_l^m(\hat{r}) &\equiv \hat{r} Y_l^m(\hat{r}) \\ \underline{B}_l^m(\hat{r}) &\equiv \nabla_1 Y_l^m(\hat{r}) = \hat{\theta} \frac{\partial Y_l^m}{\partial \theta} + \hat{\phi} \frac{im}{\sin\theta} Y_l^m \\ \underline{C}_l^m(\hat{r}) &\equiv -\hat{r} \times \nabla_1 Y_l^m(\hat{r}) = \hat{\theta} \frac{im}{\sin\theta} Y_l^m - \hat{\phi} \frac{\partial Y_l^m}{\partial \theta} \end{aligned}}$$

$\underline{B}_{lm}$ & $\underline{C}_{lm}$ normalized differently in D&T.

These called vector spherical harmonics, $\underline{P}_l^m$, $\underline{B}_l^m$, $\underline{C}_l^m$ is the notation of, e.g., Morse + Feshbach.

They are used to expand a vector field $\underline{s}$ on Ω the same way the Y_l^m's are used to expand a scalar field f on Ω.

Note $Y_0^0 = \frac{1}{\sqrt{4\pi}}$, so

$$\boxed{\begin{array}{l} \underline{P}_0^0 = \frac{1}{\sqrt{4\pi}}\,\hat{r}\,, \quad \text{but} \\ \underline{B}_0^0 = \underline{C}_0^0 = \underline{0} \quad \text{since } \nabla_1(\text{const}) = 0 \end{array}}$$

This related to the uniqueness of the expansion $\underline{t} = \nabla_1 V - \hat{r} \times \nabla_1 W$. Rendered unique by setting $V_0^0 = W_0^0 = 0$. Note any other V_0^0, W_0^0 would give same $\underline{s}$ since $\underline{B}_0^0 = \underline{C}_0^0 = \underline{0}$.

Reason for $-$ sign in $\underline{t} = \nabla_1 V - \hat{r} \times \nabla_1 W$ is to agree with Morse + Feshbach's convention for $\underline{C}_\ell^m$.

A note on expanding in vector spherical harmonics: the tangent tensor representation theorem reduces this to expanding U, V, W in Y_ℓ^m's. Thus all the Y_ℓ^m expansion theorems are applicable, e.g. if $U, V, W \in \mathcal{L}_2(\Omega)$, the Laplace series converge in the mean, and are unique.

How to find the coefficients?

The vector spherical harmonics are <u>mutually orthogonal</u>. Consider

$$\int_\Omega \underline{P}_\ell^m \cdot \underline{B}_{\ell'}^{m'*} \, dA \qquad \text{and}$$

$$\int_\Omega \underline{P}_\ell^m \cdot \underline{C}_{\ell'}^{m'*} \, dA$$

Both clearly vanish since $\hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\phi} = 0$

Now consider $\int_\Omega \underline{B}_\ell^m \cdot \underline{C}_{\ell'}^{m'*} \, dA$

$\frac{1}{\sqrt{\ell(\ell+1)\ell'(\ell'+1)}}$ with DLT normalization

$$= \frac{1}{\sqrt{\ell(\ell+1)\ell'(\ell'+1)}}\int_\Omega \left[\frac{\partial Y_\ell^m}{\partial\theta}\left(-\frac{im' Y_{\ell'}^{m'*}}{\sin\theta}\right) + \left(-\frac{im Y_\ell^m}{\sin\theta}\right)\frac{\partial Y_{\ell'}^{m'*}}{\partial\theta}\right] dA$$

For $m \neq m'$, this vanishes since

$$\int_0^{2\pi} e^{i(m-m')\phi} \, d\phi = 0$$

Consider $m = m'$. Get

$$-2\pi i m \int_0^\pi \left(\frac{\partial Y_\ell^m}{\partial\theta} Y_{\ell'}^{m*} + Y_\ell^m \frac{\partial Y_{\ell'}^{m*}}{\partial\theta}\right) d\theta$$

$$= -2\pi i m \int_0^\pi \frac{\partial}{\partial\theta}\left(Y_\ell^m Y_{\ell'}^{m*}\right) d\theta$$

$$= -2\pi i m \left[Y_\ell^m Y_{\ell'}^{m*} \right]_0^\pi = 0$$

Case $m = 0$: zero because of factor of m in front.

Case $m \neq 0$: zero because Y_ℓ^m vanishes at poles.

Now $\int_\Omega \underline{P}_\ell^m \cdot \underline{P}_{\ell'}^{m'*} \, dA$

$$= \int_\Omega Y_\ell^m Y_{\ell'}^{m'*} \, dA = \delta_{\ell\ell'} \delta_{mm'}$$

Now consider

$$\int_\Omega \underline{C}_\ell^m \cdot \underline{C}_{\ell'}^{m'*} \, dA = \frac{1}{\sqrt{\ell(\ell+1)\ell'(\ell'+1)}} \int_\Omega (-\hat{r} \times \nabla_1 Y_\ell^m) \cdot (-\hat{r} \times \nabla_1 Y_{\ell'}^{m'*}) \, dA$$

$$= \int_\Omega \nabla_1 Y_\ell^m \cdot \nabla_1 Y_{\ell'}^{m'*} \, dA = \int_\Omega \underline{B}_\ell^m \cdot \underline{B}_{\ell'}^{m'*} \, dA$$

$$= \int_\Omega \nabla_1 \cdot (Y_\ell^m \nabla_1 Y_{\ell'}^{m'*}) \, dA - \int_\Omega Y_\ell^m \nabla_1^2 Y_{\ell'}^{m'*} \, dA$$

First term vanishes by <u>Gauss' theorem</u> on Ω. Also

$$\nabla_1^2 Y_{\ell'}^{m'*} = -\ell'(\ell'+1) Y_{\ell'}^{m'*}$$

$$= l'(l'+1) \int_{\Omega} Y_l^m \, Y_{l'}^{m'*} \, dA$$

$$= l'(l'+1) \, \delta_{ll'} \, \delta_{mm'} = l(l+1) \, \delta_{ll'} \, \delta_{mm'}$$

In <u>summary</u>, we have

$$\int_{\Omega} \underline{P}_l^m \cdot \underline{P}_{l'}^{m'*} \, dA = \delta_{ll'} \, \delta_{mm'}$$

$$\int_{\Omega} \underline{B}_l^m \cdot \underline{B}_{l'}^{m'*} \, dA = \int_{\Omega} \underline{C}_l^m \cdot \underline{C}_{l'}^{m'*} \, dA = l(l+1) \, \delta_{ll'} \, \delta_{mm'}$$

$$\int_{\Omega} \underline{P}_l^m \cdot \underline{B}_{l'}^{m'*} \, dA = \int_{\Omega} \underline{P}_l^m \cdot \underline{C}_{l'}^{m'*} \, dA$$

$$= \int_{\Omega} \underline{B}_l^m \cdot \underline{C}_{l'}^{m'*} \, dA = 0.$$

Now say we have a <u>scalar</u> fcn $f(\underline{r})$; $f: \mathbb{R}^3 \to \mathbb{C}$
Then we may expand

$$\boxed{f(\underline{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m(r) \, Y_l^m(\hat{r})}$$

The coefficients f_ℓ^m are functions of radius r. They can be determined by

$$\boxed{\begin{aligned} f_\ell^m(r) &= \int_\Omega f(r\hat{r})\, Y_\ell^{m*}(\hat{r})\, dA \\ &= \int_\Omega f(r,\theta,\phi)\, Y_\ell^{m*}(\theta,\phi)\, dA \end{aligned}}$$

If we have a vector field $\underline{s}(\underline{r})$, we can expand it as

$$\boxed{\begin{aligned} \underline{s}(\underline{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \big[\, & U_\ell^m(r)\, \underline{P}_\ell^m(\hat{r}) \\ & + V_\ell^m(r)\, \underline{B}_\ell^m(\hat{r}) + W_\ell^m(r)\, \underline{C}_\ell^m(\hat{r}) \big] \end{aligned}}$$

The coefficients $U_\ell^m(r)$, $V_\ell^m(r)$, $W_\ell^m(r)$ are found by

$$\boxed{\begin{aligned} U_\ell^m(r) &= \int_\Omega \underline{s}(r\hat{r}) \cdot \underline{P}_\ell^{m*}(\hat{r})\, dA \\ \ell(\ell+1)\, V_\ell^m(r) &= \int_\Omega \underline{s}(r\hat{r}) \cdot \underline{B}_\ell^{m*}(\hat{r})\, dA \\ \ell(\ell+1)\, W_\ell^m(r) &= \int_\Omega \underline{s}(r\hat{r}) \cdot \underline{C}_\ell^{m*}(\hat{r})\, dA \end{aligned}}$$

An important property of this representation: separates curl and div in a convenient way.

Consider $\nabla \cdot \underline{s}(\underline{r})$

$$\nabla \cdot \underline{s} = \left(\hat{r}\partial_r + \frac{1}{r}\nabla_1 \right) \cdot \underline{s}$$

$$= \left(\hat{r}\partial_r + \frac{1}{r}\nabla_1 \right) \cdot \sum_l \sum_m \left(U_l^m \hat{r} Y_l^m \right.$$

$$\left. + V_l^m \nabla_1 Y_l^m - W_l^m \hat{r} \times \nabla_1 Y_l^m \right)$$

interchange ∂ and Σ.

$$= \sum_l \sum_m \Big[\partial_r U_l^m Y_l^m + \frac{1}{r} U_l^m (\nabla_1 \cdot \hat{r}) Y_l^m$$

$$+ \frac{1}{r} V_l^m \nabla_1^2 Y_l^m + \text{zero} \Big]$$

since $\underline{\nabla}_1 \cdot \underline{\Lambda} = 0$

Now $\nabla_1^2 Y_l^m = -l(l+1) Y_l^m$. What is $\nabla_1 \cdot \hat{r}$?

Consider $\nabla \cdot \underline{r} = \nabla \cdot (r\hat{r}) = \underline{\nabla r} \cdot \hat{r} + r \nabla \cdot \hat{r}$

i.e.

$$\nabla \cdot \hat{r} = \frac{1}{r}(\nabla \cdot \underline{r} - \partial_r r) = \frac{2}{r}$$

and $\nabla \cdot \hat{r} = \hat{r} \cdot \partial_r \hat{r} + \frac{1}{r} \nabla_1 \cdot \hat{r}$

$= \frac{1}{r} \nabla_1 \cdot \hat{r}$, so $\boxed{\nabla_1 \cdot \hat{r} = 2}$

And thus:

$$\boxed{\begin{aligned} \nabla\cdot \underline{s}(\underline{r}) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} D_l^m(r)\, Y_l^m(\hat{r}) \\ \text{where}\quad D_l^m(r) &= \partial_r U_l^m(r) + \tfrac{1}{r}\left[2U_l^m(r) - l(l+1)V_l^m(r)\right] \end{aligned}}$$

No contribution from $W(r)$.
Now consider $\nabla\times\underline{s}(\underline{r})$

Homework: show that

$$\begin{aligned} \nabla\times\underline{s}(\underline{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} & \left[l(l+1)\tfrac{1}{r}W_l^m(r)\right]\underline{P}_l^m(\hat{r}) \\ &+ \left[\partial_r W_l^m(r) + \tfrac{3}{r}W_l^m(r)\right]\underline{B}_l^m(\hat{r}) \\ &+ \left[\partial_r V_l^m(r) + \tfrac{1}{r}U_l^m(r) - \tfrac{1}{r}V_l^m(r)\right]\underline{C}_l^m(\hat{r}) \end{aligned}$$

<u>Nomenclature</u>: a field of the form

$-\hat{r}\times\nabla_1 W$:	toroidal
$\hat{r}U + \nabla_1 V$:	spheroidal
spheroidal <u>and</u> $\nabla\cdot\underline{s}=0$:	poloidal

These are geometrical names.

A list of properties:

Toroidal:
1. purely tangential, no $\hat{r}$ component.
2. zero divergence
3. curl is spheroidal, radial component of curl non-zero

Spheroidal:
1. has a radial component and a tangential component
2. non-zero divergence
3. curl is toroidal, has ~~a~~ no radial component.

This designation leads to the designation toroidal and spheroidal free oscillations.

31. The radial SNREI equations

Back to modes, at last. The linearized SNREI equations are

$$-\rho_0 \omega^2 \underline{s} = -\rho_0 \nabla\phi_1 - \rho_1 \nabla\phi_0 - \nabla[\rho_0 \underline{s}\cdot\nabla\phi_0] + \nabla\cdot\underline{\underline{T}}$$

$$\rho_1 = -\nabla\cdot(\rho_0 \underline{s})$$