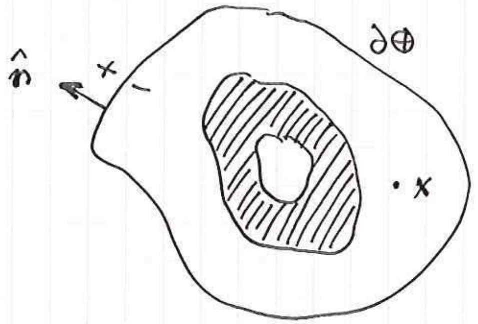
5th class
Thurs 22 Feb

Free Oscillations and Surface Waves

Consider an Earth model with a solid mantle & crust, a fluid outer core and a solid inner core



$\oplus_S$: solid regions

$\oplus_F$: fluid regions

$$\oplus = \oplus_S \cup \oplus_F$$

$$\Sigma = \Sigma_{SS} + \Sigma_{FS} + \partial\oplus$$

solid–solid e.g. Moho

fluid–solid e.g. CMB

We ignore the Earth's self-gravitation, even though it is quantitatively important for the low-frequency normal modes.

Elastodynamic equation of motion: $\rho\,\partial_t^2 \mathbf{s} = \nabla\cdot\mathbf{T}$

Hooke's law: $\mathbf{T} = \mathbf{C} : \boldsymbol{\varepsilon}$ or $T_{ij} = c_{ijkl}\,\varepsilon_{kl}$

where $\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{s} + (\nabla\mathbf{s})^T\right]$ or $\varepsilon_{ij} = \frac{1}{2}(\partial_i s_j + \partial_j s_i)$

is the elastic strain.

The tensor c_{ijkl} is the elastic tensor:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$$

$\mathbf{T}^T = \mathbf{T}$

$\boldsymbol{\varepsilon}^T = \boldsymbol{\varepsilon}$

hyperelastic material

A general anisotropic material (triclinic xtal) has 21 independent elastic components.

Isotropic material: $c_{ijkl} = (\kappa - \frac{2}{3}\mu)\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$$\mathbf{T} = (\kappa - \tfrac{2}{3}\mu)(\nabla\cdot\mathbf{s})\,\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \text{ or}$$

$$\mathbf{T} = \kappa(\nabla\cdot\mathbf{s})\,\mathbf{I} + 2\mu\mathbf{d}$$ where

$\delta = \varepsilon - \frac{1}{3}(\mathrm{tr}\,\varepsilon)\mathbf{I} = \varepsilon - \frac{1}{3}(\nabla\cdot s)\mathbf{I}$ is the deviatoric [redacted] strain ($\mathrm{tr}\,\delta = 0$)

$\mathbf{T} =$ [redacted] $\kappa(\nabla\cdot s)\mathbf{I} + 2\mu\delta$

isotropic stress $\nearrow$ (under $\kappa(\nabla\cdot s)\mathbf{I}$) ; $\uparrow$ deviatoric stress (under $2\mu\delta$)

κ – bulk modulus or incompressibility
μ – shear modulus or rigidity

Fluid region: $\mu = 0$ (no rigidity)

$$\mathbf{T} = \kappa(\nabla\cdot s)\mathbf{I} \quad \text{in } \oplus_F$$

Boundary conditions:

kinematic: $[s]^+_- = 0$ on Σ_{SS} (welded)

$[\hat{n}\cdot s]^+_- = 0$ on Σ_{FS} (tangential slip allowed)

dynamic: $[\hat{n}\cdot\mathbf{T}]^+_- = 0$ traction continuous on both Σ_{SS} & Σ_{FS}

$\hat{n}\cdot\mathbf{T} = 0$ on $\partial\oplus$: free outer surface

Conservation of energy:

Consider $\displaystyle\int_{\oplus} \partial_t s \cdot (\rho\,\partial_t^2 s - \nabla\cdot\mathbf{T})\, dV$

First term is $\displaystyle\frac{d}{dt}\int_{\oplus} \frac{1}{2}\rho\,|\partial_t s|^2\, dV$

[redacted]

[redacted]

Second term is $-\int_{\oplus} \partial_t s_j \, \partial_i T_{ij} \, dV$

$$= \int_{\Sigma} \cancel{[\partial_t s_j \, n_i T_{ij}]^{+}_{-}} \, d\Sigma + \int_{\oplus} T_{ij} \, \partial_t (\partial_i s_j) \, dV$$

→ vanishes by b.c.

why: if clause in Gauss' theorem

(note — must apply Gauss' theorem separately to each sub-region)

$$= \int_{\oplus} c_{ijkl} \, \partial_k s_l \, \partial_t (\partial_i s_j) \, dV$$

$$= \frac{d}{dt} \int_{\oplus} \tfrac{1}{2} c_{ijkl} \, \partial_i s_j \, \partial_k s_l \, dV$$

since $c_{ijkl} = c_{klij}$

In summary:

$$\frac{d}{dt} \int_{\oplus} \left[\tfrac{1}{2} \rho |\partial_t s|^2 + \tfrac{1}{2} c_{ijkl} \, \partial_i s_j \, \partial_k s_l \right] dV = 0$$

$\frac{1}{2} \rho |\partial_t s|^2$: kinetic energy density

$\frac{1}{2} c_{ijkl} \, \varepsilon_{ij} \, \varepsilon_{kl}$: stored elastic potential energy density

The sum of the kinetic + potential energy is <u>conserved</u> : $\frac{d}{dt}(T + V) = 0$.

In an isotropic material, the elastic energy density is

$$\underbrace{\tfrac{1}{2} \kappa (\nabla \cdot s)^2}_{\text{compressional energy density}} + \underbrace{\mu \, \delta : \delta}_{\text{shear energy density}}$$

stability: $\kappa > 0$, $\mu > 0$

more generally **c** is positive definite

every deformation requires work

A hyperelastic material ($c_{ijkl} = c_{klij}$) is one having an elastic energy density.

Hamilton's principle

The equations of motion and boundary conditions can be derived from a variational principle. Define the action integral

$$I = \int_{t_1}^{t_2} \int_{\oplus} L \, dV \, dt \qquad L(\overset{\text{not really}\,\downarrow}{s}, \partial_t s, \nabla s)$$

where $L = \frac{1}{2}\rho|\partial_t s|^2 - \frac{1}{2} C_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl}$ is the Lagrangian density

kinetic − potential

We regard I as a functional of the dynamical path $s(x,t)$ between times t_1 and t_2. Consider the variation of this functional for fixed end-points $\delta s(x,t_1) = \delta s(x,t_2) = 0$. A variation δs is admissible if $[\delta s]^{\pm} = 0$ on Σ_{SS} and $[\hat{n}\cdot\delta s]^{\pm} = 0$ on Σ_{FS}.

$$\delta I = \int_{t_1}^{t_2} \int_{\oplus} [\, \delta s \cdot \partial_s L + \partial_t(\delta s) \cdot \partial_{\partial_t s} L + \nabla(\delta s) \cdot \partial_{\nabla s} L \,] \, dV \, dt$$

$$= \int_{t_1}^{t_2} \int_{\oplus} \delta s \cdot \left[\partial_s L - \partial_t (\partial_{\partial_t s} L) - \nabla \cdot (\partial_{\nabla s} L) \right] dV \, dt$$

$$+ \int_{\oplus} \Big[\underbrace{\delta s \cdot \partial_{\partial_t s} L}_{\text{zero since ends of path are fixed}} \Big]_{t_1}^{t_2} dV$$

$$- \int_{t_1}^{t_2} \int_{\Sigma} [\, \delta s \cdot (\hat{n} \cdot \partial_{\nabla s} L) \,]^{\pm} \, d\Sigma \, dt$$

δI vanishes for an arbitrary admissible δs if and only if

$$\partial_s L - \partial_t(\partial_{\partial_t s} L) - \nabla\cdot(\partial_{\nabla s} L) = 0 \quad \text{in } \oplus$$

Euler – Lagrange eqn

$$[\hat{n}\cdot(\partial_{\nabla s} L)]^{+}_{-} \quad \text{on } \Sigma$$

$\partial_s L = 0$

$\partial_{\partial_t s} L = \rho\, \partial_t s$ (momentum density)

$\partial_{\nabla s} L = -\mathbf{T}$ (– stress)

Euler – Lagrange equation : $\rho\, \partial_t^2 s = \nabla\cdot\mathbf{T}$

Boundary condition : $[\hat{n}\cdot\mathbf{T}]^{+}_{-} = 0$

The underlined term is $-\int_{t_1}^{t_2}\int_{\Sigma} [\delta s\cdot(\hat{n}\cdot\mathbf{T})]^{+}_{-}\, d\Sigma\, dt$

end here 6th class

on Σ_{SS} : $[\delta s]^{+}_{-} = 0$

on Σ_{FS} : $[\hat{n}\cdot\delta s]^{+}_{-} = 0$ but $\hat{n}\cdot\mathbf{T} = \hat{n}(\hat{n}\cdot\mathbf{T})$

dynamical b.c. arise naturally   no shear stress on Σ_{FS}.

Normal mode solutions : oscillatory in time

Ch. IV of D&T.

$$s(x,t) = s(x)\begin{Bmatrix}\sin\omega t\\ \cos\omega t\end{Bmatrix}$$

why? because $\rho(x)$ and $C(x)$ ind. of time t

today Sect. 4.1

$s(x)$: real eigenfunction

ω : real eigenfrequency

$$-\rho\omega^2 s = \nabla\cdot\mathbf{T}$$

dependence on ω^2 ⇒ two eigenfrequencies $\pm\omega$ associated with every s.

Write in abstract operator notation :

$Hs = \omega^2 s$ where $Hs = -\frac{1}{\rho}\nabla\cdot(\mathbf{C}:\nabla s)$

or $Hs_j = -\frac{1}{\rho}\partial_i(C_{ijkl}\,\partial_k s_l)$

H stands for the differential operator <u>together with the b.c.</u>

Eigenvalue problem : $Hs = \overset{\text{eigenvalue}}{\overset{\downarrow}{\omega^2}} s$

Define the <u>inner product</u> of two functions s and s':

$$\langle s, s'\rangle = \int_\oplus \underset{\uparrow\ \text{density weighting}}{\rho}\, s\cdot s'\, dV$$

The operator H is <u>Hermitian</u> or self-<u>adjoint</u> with respect to $\langle\cdot,\cdot\rangle$, i.e.

$$\langle s, Hs'\rangle = \langle Hs, s'\rangle = \langle s', Hs\rangle$$

Proof:

$$\langle s, Hs'\rangle = -\int_\oplus s\cdot(\nabla\cdot\mathbf{T}')\,dV$$

$$= \int_\oplus C_{ijkl}\,\partial_i s_j\,\partial_k s'_l\,dV$$

$$+ \int_\Sigma [\hat{n}\cdot\mathbf{T}'\cdot s]_-^+\,d\Sigma$$

equal since $C_{ijkl} = C_{klij}$

$$\langle s', Hs\rangle = -\int_\oplus s'\cdot(\nabla\cdot\mathbf{T})\,dV$$

$$= \int_\oplus C_{ijkl}\,\partial_i s'_j\,\partial_k s_l\,dV$$

$$+ \int_\Sigma [\hat{n}\cdot\mathbf{T}\cdot s']_-^+\,d\Sigma$$

The surface integrals vanish by virtue of the b.c. (this is why H "includes the b.c.")

It is noteworthy that the manipulations required to show that H is Hermitian are the same as those used to show $\frac{d}{dt}(T+V)=0$. Illustrates a general principle — that physical systems governed by Hermitian operators are energy-conserving.

Consider now the inner product of $Hs=\omega^2 s$ with s' and the inner product of $Hs'=\omega'^2 s'$ with s:

$$\omega^2\langle s', s\rangle = \langle s', Hs\rangle$$
$$\omega'^2\langle s, s'\rangle = \langle s, Hs'\rangle$$

Subtracting and using the Hermitsicity we find

$$(\omega^2-\omega'^2)\langle s, s'\rangle = 0 \quad \text{or}$$

$$\langle s, s'\rangle = 0 \quad \text{if} \quad \omega^2 \neq \omega'^2$$

Eigenfunctions associated with discrete eigenfrequencies $\omega \neq \omega'$ are <u>orthogonal</u> in the sense $\langle s, s'\rangle = 0$. Because of this every eigensolution ω, s is referred to as a normal mode.

<u>Rayleigh's principle</u>:

Every Hermitian eigenvalue problem of the form $Hs = \omega^2 s$ is associated with a variational principle known as Rayleigh's principle.

Consider the Rayleigh quotient

$$\omega^2 = \frac{\langle s, \mathcal{H}s \rangle}{\langle s, s \rangle}$$

Regard right side as a functional which assigns a scalar $\omega^2(s)$ to every possible displacement s. Then Rayleigh's principle asserts that this functional is stationary for every admissible δs iff s is an eigenfunction with associated squared eigenfrequency ω^2. To verify this,

$$\delta\omega^2 = \frac{\langle \delta s, \mathcal{H}s \rangle + \langle s, \mathcal{H}\delta s \rangle - \omega^2 \langle \delta s, s \rangle - \omega^2 \langle s, \delta s \rangle}{\langle s, s \rangle}$$

$$= \frac{2 \langle \delta s, \mathcal{H}s - \omega^2 s \rangle}{\langle s, s \rangle}$$

Evidently $\delta\omega^2 = 0$ for an arbitrary δs iff $\mathcal{H}s = \omega^2 s$.

We may alternatively consider the quantity

$$\mathcal{L} = \tfrac{1}{2}\omega^2 \langle s, s \rangle - \tfrac{1}{2}\langle s, \mathcal{H}s \rangle$$

rather than ω^2 to be the stationary functional.

An alternative notation – define the kinetic and potential energy quadratic functionals

$$\mathcal{T} = \int_{\oplus} \rho\, s \cdot s \, dV$$

$$\mathcal{V} = \int_{\oplus} C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} \, dV.$$

Then $$\omega^2 = \frac{\mathcal{V}}{\mathcal{T}} = \frac{\text{potential energy}}{\text{kinetic energy}}$$

$$\mathcal{L} = \tfrac{1}{2}(\omega^2\mathcal{T} - \mathcal{V}) = \text{kinetic energy} - \text{potential energy}$$

Fleshing out the above schematic proof:

$$\delta\mathcal{L} = \int_{\oplus} \delta s \cdot [\,\omega^2 \rho s + \nabla\cdot\mathbf{T}\,]\, dV$$
$$+ \int_{\Sigma} [\,\delta s \cdot (\hat{n}\cdot\mathbf{T})\,]^{\pm}_{-}\, d\Sigma$$

Clearly $\delta\mathcal{L}$ vanishes for an arbitrary δs iff

$$-\rho\omega^2 s = \nabla\cdot\mathbf{T} \quad \text{in } \oplus$$
$$[\hat{n}\cdot\mathbf{T}]^{\pm}_{-} = 0 \quad \text{on } \Sigma$$

The stationary value of $\mathcal{L}$ at the eigensolutions ω^2, s is $\mathcal{L} = 0$. Physically, $\omega^2\mathcal{T}$ and $\mathcal{V}$ are twice the average k.e. and p.e. during a cycle of free oscillation $s(x)\cos\omega t$ or $s(x)\sin\omega t$.

Trivial modes:

Every Earth model has a 6-dimensional space of trivial rigid-body modes

$$\omega^2 = 0, \quad s(x) = \underset{\text{3 translations}}{\mathbf{X}} + \underset{\text{3 rotations}}{\mathbf{Q}\cdot\mathbf{x}}$$

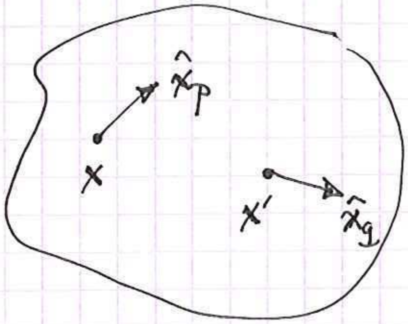
In addition there is an ∞-dimensional family of trivial geostrophic ($\omega^2 = 0$) modes confined to the fluid core.

Go to page 14.2: do Rayleigh's principle before this.

Green's tensor

The response to an earthquake, meteorite impact, nuclear explosion, etc. can be conveniently expressed in terms of the Green's tensor or impulse response:

Definition: $G_{pq}(x,x';t)$ is the $\hat{x}_p$ component of the response at x,t to a unit impulsive force [redacted] in the $\hat{x}_q$ direction at $x',0$.



In other words: $\rho(\partial_t^2 G + \mathcal{H}G) = \mathbb{I}\,\delta(x-x')\,\delta(t)$

Equivalently, we can solve the homogeneous eqn

$$\rho(\partial_t^2 G + \mathcal{H}G) = 0 \quad , \quad t \geq 0$$

subject to the initial conditions

$$G(x,x';0) = 0$$

$$\partial_t G(x,x';0) = \rho^{-1}\,\mathbb{I}\,\delta(x-x')$$

Label the eigensolutions $\pm\omega_k$, s_k with an index k and normalize such that

$$\langle s_k, s_{k'}\rangle = \int_{\oplus} \rho\, s_k \cdot s_{k'}\, dV = \delta_{kk'}$$

Look for a solution that is a linear combination of free oscillations (assume the s_k are <u>complete</u>), for $t \geq 0$:

$$G(x,x';t) = \sum_k s_k(x) \left[a_k(x') \cos\omega_k t + b_k(x') \sin\omega_k t \right] H(t)$$

This satisfies $\rho(\partial_t^2 G + \mathcal{H} G) = 0$.
The b.c. are satisfied if

$$\sum_k s_k a_k = 0 \quad ; \quad \sum_k \omega_k s_k b_k = \rho^{-1} \mathbb{I}\, \delta(x-x')$$

Take inner product with s_k and use orthonormality:

$$a_k = 0 \quad ; \quad b_k = \omega_k^{-1} s_k(x')$$

Thus the normal-mode Green's tensor is

$$G(x,x';t) = \sum_k \omega_k^{-1} s_k(x)\, s_k(x') \sin\omega_k t \, H(t)$$

Every mode begins oscillating with the same phase, like $\sin\omega_k t$.
Note that

$$G(x,x';t) = G^T(x',x;t)$$

$G_{pq}(x,x';t)$
$= G_{qp}(x',x;t)$

This is the principle of source-receiver <u>reciprocity</u>. Note that the <u>orientations</u> of source and receiver as well as their <u>locations</u> must be interchanged.

Response to a transient force:

The response to a general applied body force f in $\oplus$ and surface force t on $\partial\oplus$ can be found by convolution with the impulse response:

$$s(x,t) = \int_{-\infty}^{t} \int_{\oplus} G(x,x';t-t') \cdot f(x',t')\, dV'\, dt'$$
$$+ \int_{-\infty}^{t} \int_{\partial\oplus} G(x,x';t-t') \cdot t(x',t')\, d\Sigma'\, dt'$$

Superposition plus causality (upper limit is t). Lower limit can be any time before f and t begin to act (entire past history).

Substituting the mode-sum Green's tensor gives

$$s(x,t) = \sum_k \omega_k^{-1} s_k(x) \int_{-\infty}^{t} A_k(t') \underbrace{\sin\omega_k(t-t')}_{\text{impulse response}} dt'$$

where

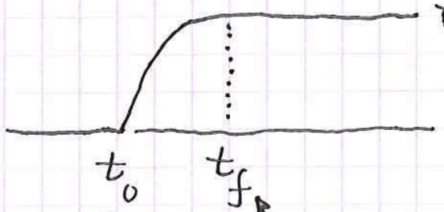
$$A_k(t) = \int_{\oplus} f(x,t) \cdot s_k(x)\, dV + \int_{\partial\oplus} t(x,t) \cdot s_k(x)\, d\Sigma$$

projection onto the mode

Integrating by parts with respect to time, can also write in form

$$s(x,t) = \sum_k \omega_k^{-2} s_k(x) \int_{-\infty}^{t} \partial_{t'} A_k(t') \underbrace{\left[1 - \cos\omega_k (t - t') \right]}_{\text{unit step response}} dt'$$

As we shall see later, the *equivalent* forces to an earthquake have the character



$\mathbf{f}_{static}$, $\mathbf{t}_{static}$ ← correspond to final *static* slip on fault

origin time

active faulting ceases

The responce to such a *transient* force, for times $t \geq t_f$, is

$$s(x,t) = \sum_k \omega_k^{-2} \left(a_k^f - a_k \cos\omega_k t - b_k \sin\omega_k t \right) s_k(x) \quad , \quad t \geq t_f$$

$$a_k^{stat} = \int_{\oplus} \mathbf{f}_{stat} \cdot s_k \, dV + \int_{\partial\oplus} \mathbf{t}_{stat} \cdot s_k \, d\Sigma$$

$$a_k = \int_{t_0}^{t_f} \int_{\oplus} \partial_t \mathbf{f} \cdot s_k \cos\omega_k t \, dV \, dt + \int_{t_0}^{t_f} \partial_t \mathbf{t} \cdot s_k \cos\omega_k t \, d\Sigma \, dt$$

$$b_k = \int_{t_0}^{t_f} \int_{\oplus} \partial_t \mathbf{f} \cdot s_k \sin\omega_k t \, dV \, dt + \int_{t_0}^{t_f} \partial_t \mathbf{t} \cdot s_k \sin\omega_k t \, d\Sigma \, dt$$

integrals only over active faulting interval

Anelasticity causes the oscillations $\cos\omega_k t$, $\sin\omega_k t$ to decay with time:

$$\lim_{t\to\infty} s(x,t) \equiv s_{stat}(x) = \sum_k \omega_k^{-2} a_k^{stat} s_k(x)$$

This represents the permanent static deformation of the Earth produced by the faulting.

Modern seismic instruments are in essence accelerometers — the acceleration is

$$a(x,t) = \sum_k \underbrace{(a_k \cos\omega_k t + b_k \sin\omega_k t)}_{\text{free oscillations}} s_k(x) \quad , \quad t \geq t_f$$

$s_k(x)$ is the geographic shape of the k^{th} oscillation

Every detail of every observed seismogram is of this form (with attenuation properly accounted for).

[redacted]

Rayleigh – Ritz method :

Let $\mathbf{e}_k$ be a set of functions in $\oplus$, smooth everywhere except on Σ_{FS}, where they satisfy $[\hat{n}\cdot\mathbf{e}_k]=0$. For now the $\mathbf{e}_k$ can be considered arbitrary — later we will take them to be the eigenfunctions of a starting Earth model.

Write eigenfunction s as an expansion

$$s=\sum_k a_k\mathbf{e}_k$$

Substitute into the action $\mathcal{L}=\frac{1}{2}(\omega^2\mathcal{T}-\mathcal{V})$

Get new form for action:

$$\mathcal{L}=\frac{1}{2}\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\mathbf{a}$$

where

$$\mathbf{a}=\begin{pmatrix}\vdots\\ a_k\\ \vdots\end{pmatrix},\quad \mathbf{T}=\begin{pmatrix} & \vdots & \\ \cdots & T_{kk'} & \cdots\\ & \vdots & \end{pmatrix},\quad \mathbf{V}=\begin{pmatrix} & \vdots & \\ \cdots & V_{kk'} & \cdots\\ & \vdots & \end{pmatrix}$$

$$T_{kk'}=\int_{\oplus}\rho\,\mathbf{e}_k\cdot\mathbf{e}_{k'}\,dV \qquad V_{kk'}=\int_{\oplus}\boldsymbol{\varepsilon}_k:\mathbf{C}:\boldsymbol{\varepsilon}_{k'}\,dV$$

$$=\int_{\oplus}[\kappa(\nabla\cdot\mathbf{e}_k)(\nabla\cdot\mathbf{e}_{k'})+2\mu\,\boldsymbol{\delta}_k:\boldsymbol{\delta}_{k'}]\,dV$$

$\mathbf{T}$ and $\mathbf{V}$ are symmetric matrices : $\mathbf{T}^T=\mathbf{T}$, $\mathbf{V}^T=\mathbf{V}$

The variation of the action with respect to $\mathbf{a}$ is

$$\delta\mathcal{L}=\frac{1}{2}\delta\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\mathbf{a}+\frac{1}{2}\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\delta\mathbf{a}$$

$$=\delta\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\mathbf{a}=0 \quad \text{iff} \quad \mathbf{V}\cdot\mathbf{a}=\omega^2\mathbf{T}\cdot\mathbf{a}$$

$\mathbf{T}$ is symmetric and positive definite $\mathbf{T}>0$ so it can be inverted : $(\mathbf{T}^{-1}\cdot\mathbf{V})\cdot\mathbf{a}=\omega^2\mathbf{a}$, an ordinary matrix eigenvalue problem. No need to solve ODE's — only need to do matrix algebra.

14.2

Rayleigh's principle can also be used to calculate the change in an eigenfrequency ω due to a change in the Earth model:

$$\rho \to \rho + \delta\rho \,, \quad \kappa \to \kappa + \delta\kappa \,, \quad \mu \to \mu + \delta\mu$$

as a result $\omega \to \omega + \delta\omega \,,\ s + \delta s$

↑ no need to calculate

We regard the Lagrangian density L as a functional not only of s and ∇s, but also of ω and $\oplus \equiv \{\rho, \kappa, \mu\}$. Write the action as

$$\mathcal{A} = \int_\oplus L(s, \nabla s, \omega, \oplus)\, dV \quad \text{where}$$

$$L = \tfrac{1}{2}\left[\omega^2 \rho\, s\cdot s - \kappa(\nabla\cdot s)^2 - 2\mu\, \delta : \delta\right]$$

do this in terms of C_{ijkl}

We know 2 things about $\mathcal{A}$:

(1) it is stationary

(2) it's value at the stationary points (eigensolutions) is zero

Consider $\mathcal{A} = \int_\oplus L(s, \nabla s, \omega^2, \oplus)\, dV = 0$

Take the <u>total</u> variation w.r.t. $s, \nabla s, \omega^2, \oplus$

$$\int_\oplus \left[\delta s \cdot \partial_s L + \nabla s : \partial_{\nabla s} L + \delta\omega^2\, \partial_{\omega^2} L + \delta\oplus\, \partial_\oplus L\right] dV = 0$$

$$\int_\oplus \delta s \cdot \underbrace{\left[\partial_s L - \nabla\cdot(\partial_{\nabla s} L)\right]}_{\rho\omega^2 s + \nabla\cdot T = 0} dV - \int_\Sigma \left[\delta s \cdot \underbrace{(\hat{n}\cdot\partial_{\nabla s} L)}_{-\hat{n}\cdot T = 0}\right]^{\pm} d\Sigma$$

$$+ \int_\oplus \left[\delta\omega^2\, \partial_{\omega^2} L + \delta\oplus\, \partial_\oplus L\right] dV = 0$$

$$\delta\omega^2 \int_\oplus \partial_\omega^2 L \, dV = - \int_\oplus \delta\oplus \, \partial_\oplus L$$

write in terms of δC_{ijkl}

$$= - \int_\oplus [\, \delta\rho \, \partial_\rho L + \delta\kappa \, \partial_\kappa L + \delta\mu \, \partial_\mu L \,] \, dV$$

$\partial_\omega^2 L = \frac{1}{2} \rho s \cdot s$

If we adopt the normalization $\int_\oplus \rho s \cdot s \, dV = 1$:

$$\delta\omega^2 = \int_\oplus [\, \delta\kappa \, (\nabla \cdot s)^2 + 2 \delta\mu \left(\mathbf{d} : \mathbf{d} \right) - \delta\rho \, \omega^2 s \cdot s \,] \, dV$$

$\underline{\varepsilon} : \delta C : \varepsilon$

$$\delta\omega^2 = \int_\oplus [\, \delta\kappa \, (\nabla \cdot s)^2 + 2 \mu \, (\mathbf{d} : \mathbf{d}) - \delta\rho \, \omega^2 s \cdot s \,] \, dV$$

This is the basis for SNREI Earth model inversion. One measures the observed frequencies of vibration after an earthquake and calculates the residuals $\delta\omega = \omega_{obs} - \omega_{model}$ relative to some starting or initial model. The above result can then be used to adjust the model to provide a best fit to the data.

Quasi-degenerate perturbation theory

The above only works if the initial eigenfrequencies are well isolated in the spectrum. More generally we can seek to find all the zeroth-order eigenfunctions s and first-order

eigenfrequencies $\omega_0 + \delta\omega$ in the vicinity of some fiducial or reference frequency ω_0.
Use the Rayleigh-Ritz form of the action but now choose the basis vectors $\mathbf{e}_k$ to be the eigenfunctions of the initial model: $\mathbf{e}_k = \mathbf{s}_k$. The kinetic and potential energy matrices now take the form

$$\mathbf{T} = \mathbf{I} + \delta\mathbf{T}$$

$$\mathbf{V} = \mathbf{\Omega}^2 + \delta\mathbf{V} \qquad \text{where} \qquad \mathbf{\Omega} = \begin{pmatrix} \ddots & & \\ & \omega_k & \\ & & \ddots \end{pmatrix}$$

Reason: $\quad T_{kk'} = \int_{\oplus} (\rho + \delta\rho)\, \mathbf{s}_k \cdot \mathbf{s}_k \, dV$, etc.

Substitute in $\mathbf{V}\cdot\mathbf{a} = \omega^2\, \mathbf{T}\cdot\mathbf{a}$ and neglect second-order terms

$$(\mathbf{\Omega}^2 + \delta\mathbf{V})\cdot\mathbf{a} = (\omega_0^2 + 2\omega_0\,\delta\omega)(\mathbf{I} + \delta\mathbf{T})\cdot\mathbf{a}$$

$$(\mathbf{\Omega}^2 - \omega_0^2\,\mathbf{I} + \delta\mathbf{V} - \omega_0^2\,\delta\mathbf{T})\cdot\mathbf{a} = 2\omega_0\,\delta\omega\,\mathbf{a}$$

$$\mathbf{H}\cdot\mathbf{a} = \delta\omega\,\mathbf{a} \leftarrow \text{an } N\times N \text{ algebraic eigenvalue problem}$$

$$\mathbf{H} = \frac{1}{2\omega_0}(\mathbf{\Omega}^2 - \omega_0^2\,\mathbf{I} + \delta\mathbf{V} - \omega_0^2\,\delta\mathbf{T})$$

$$\mathbf{H} = \frac{1}{2\omega_0}\underbrace{\begin{pmatrix} \ddots & & \\ & \omega_k^2 - \omega_0^2 & \\ & & \ddots \end{pmatrix}}_{\text{diagonal}} + \frac{1}{2\omega_0}\underbrace{\begin{pmatrix} & \vdots & \\ \cdots & \delta V_{kk'} - \omega_0^2\,\delta T_{kk'} & \cdots \\ & \vdots & \end{pmatrix}}_{\text{symmetric}}$$

The zeroth-order eigenfunctions of the perturbed model are $\mathbf{s} = \sum_k a_k \mathbf{s}_k$ and the associated first-order eigenfrequencies are $\omega_0 + \delta\omega$

Seismic source representation :

begin class #8
Ch. 5 D&T

The simplest and most general approach to this problem uses the concept of the stress glut, introduced by Backus & Mulcahy in 1976.

We have used two equations

$$\rho \partial_t^2 s = \nabla \cdot \mathbf{T} \quad : \quad \text{Newton's second law}$$

$$\mathbf{T} = \mathbf{C} : \boldsymbol{\varepsilon} \quad : \quad \text{Hooke's "law"} \qquad \text{b.c. } \hat{n} \cdot \mathbf{T} = 0 \quad \text{on } \partial\oplus$$

The first is a bona fide (linearized) law of physics — the second is not. If both "laws" were always valid there would be no earthquakes — the equations are homogeneous.

Every indigenous source, which does not involve forces exerted by other bodies (e.g. a meterite strike) are the result of a localized, transient failure of Hooke's law (including slip on faults, phase changes, etc.)

We regard Newton's law and the b.c. as true for the true physical stress $\mathbf{T}_{\text{true}}$:

$$\rho \partial_t^2 s = \nabla \cdot \mathbf{T}_{\text{true}} \quad \text{in } \oplus$$

$$\hat{n} \cdot \mathbf{T}_{\text{true}} = 0 \quad \text{on } \partial\oplus$$

But Hooke's law only defines the Hooke stress

$$\mathbf{T}_{\text{Hooke}} = \mathbf{C} : \boldsymbol{\varepsilon}$$

Define the Backus–Mulcahy stress glut by

$$\mathbf{T}_{glut} \equiv \mathbf{T}_{Hooke} - \mathbf{T}_{true}$$ D&T call the glut $\mathbf{S}$

Rewrite eqn & b.c. in form

$$\rho \partial_t^2 s = \nabla \cdot \mathbf{T}_{Hooke} - \nabla \cdot \mathbf{T}_{glut} \quad \text{in } \oplus$$

$$\hat{n} \cdot \mathbf{T}_{Hooke} = \hat{n} \cdot \mathbf{T}_{glut} \quad \text{on } \partial\oplus$$

Define the equivalent body and surface forces

$$f \equiv -\nabla \cdot \mathbf{T}_{glut} \quad \text{in } \oplus$$

$$t \equiv \hat{n} \cdot \mathbf{T}_{glut} \quad \text{on } \partial\oplus$$

Then have a non-homogeneous problem

$$\left.\begin{aligned} \rho \partial_t^2 s &= \nabla \cdot \mathbf{T} + f \quad \text{in } \oplus \\ \mathbf{T} &= \mathbf{C} : \varepsilon \quad \text{in } \oplus \\ \hat{n} \cdot \mathbf{T} &= t \quad \text{on } \partial\oplus \end{aligned}\right\}$$

where $\mathbf{T}$ denotes $\mathbf{T}_{Hooke}$ – the linear model stress

f and t act as sources that can excite the free oscillations of the Earth.

[redacted]

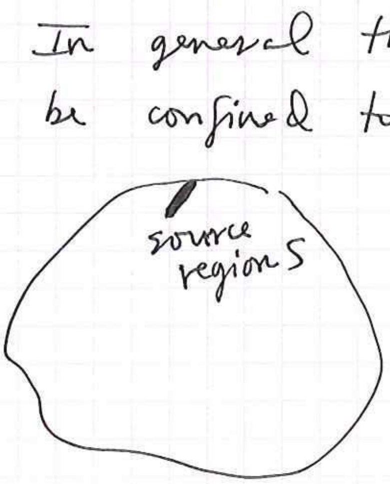
In general the breakdown of Hooke's law will be confined to some source region S

$\mathbf{T}_{glut} = 0$ in $\oplus - S$ and on $\partial S - \partial S \cap \partial\oplus$

$f = 0$ in $\oplus - S$

$t = 0$ on $\underbrace{\partial S - \partial S \cap \partial\oplus}_{\text{fault may intersect surface (as shown)}}$

this eqn not right
t only defined on $\partial\oplus$



The total force exerted on the Earth by $\mathbf{f}$ and $\mathbf{t}$ is

$$\mathbf{F}_{total} = \int_{\oplus} \mathbf{f}\, dV + \int_{\partial\oplus} \mathbf{t}\, d\Sigma$$

$$= -\int_{S} \nabla\cdot\mathbf{T}_{glut}\, dV + \int_{\partial S \cap \partial\oplus} \hat{\mathbf{n}}\cdot\mathbf{T}_{glut}\, d\Sigma$$

$$= -\int_{\partial S - \partial S \cap \partial\oplus} \hat{\mathbf{n}}\cdot\mathbf{T}_{glut}\, d\Sigma = 0$$

The total torque likewise vanishes:

$$\mathbf{N}_{total} = \int_{\oplus} \mathbf{x}\times\mathbf{f}\, dV + \int_{\partial\oplus} \mathbf{x}\times\mathbf{t}\, d\Sigma$$

$$= -\int_{S} \mathbf{x}\times(\nabla\cdot\mathbf{T}_{glut})\, dV + \int_{\partial S\cap\partial\oplus} \mathbf{x}\times(\hat{\mathbf{n}}\cdot\mathbf{T}_{glut})\, d\Sigma$$

The ith component is

$$-\int_{S} \varepsilon_{ijk}\, x_j\, \partial_l T_{lk}^{glut}\, dV + \int_{\partial S\cap\partial\oplus} \varepsilon_{ijk}\, x_j\, n_l T_{lk}^{glut}\, d\Sigma$$

$\varepsilon_{ilk} T_{lk}^{glut}$

$$= \int_{S} \underbrace{\varepsilon_{ijk} T_{jk}^{glut}}_{\substack{\text{vanishes}\\ \text{since}\\ T_{jk}^{glut} = T_{kj}^{glut}.}}\, dV - \int_{\partial S - \partial S\cap\partial\oplus} \varepsilon_{ijk}\, x_j\, n_l T_{lk}^{glut}\, d\Sigma = 0$$

An indigenous source exerts no net force or torque on the Earth — the trivial modes are not excited as a result.

Recall the acceleration response to a transient applied $\mathbf{f}$ and $\mathbf{t}$:

$$\mathbf{a}\,\cancel{\mathbf{s}}(\mathbf{x},t) = \sum_k \left(a_k \cos\omega_k t + b_k \sin\omega_k t\right) \mathbf{s}_k(\mathbf{x})$$

What are the excitation amplitudes for a stress-glut source?

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \int_{t_0}^{t_f} \underbrace{\left[\int_{\oplus} \partial_t \mathbf{f} \cdot \mathbf{s}_k \, dV + \int_{\partial\oplus} \partial_t \mathbf{t} \cdot \mathbf{s}_k \, d\Sigma \right] \begin{Bmatrix} \cos\omega_k t \\ \sin\omega_k t \end{Bmatrix}}_{} dt$$

$$-\int_{S} \nabla \cdot \partial_t \mathbf{T}_{glut} \cdot \mathbf{s}_k \, dV + \int_{\partial S \cap \partial\oplus} \hat{\mathbf{n}} \cdot \partial_t \mathbf{T}_{glut} \cdot \mathbf{s}_k \, d\Sigma$$

$$= \int_S \partial_t \mathbf{T}_{glut} : \nabla \mathbf{s}_k \, dV - \int_{\partial S - \partial S \cap \partial\oplus} \hat{\mathbf{n}} \cdot \partial_t \mathbf{T}_{glut} \cdot \mathbf{s}_k \, d\Sigma$$

$$= \int_S \partial_t \mathbf{T}_{glut} : \boldsymbol{\varepsilon}_k \, dV$$

↑ strain associated with k^{th} eigenfunction

integral ↑ only over source volume

$$\boldsymbol{\varepsilon}_k = \tfrac{1}{2}\left[\nabla \mathbf{s}_k + (\nabla \mathbf{s}_k)^T\right]$$

In summary:

$$a_k = \int_{t_0}^{t_f} \int_S \partial_t \mathbf{T}_{glut} : \boldsymbol{\varepsilon}_k \, dV \cos\omega_k t \, dt$$

$$b_k = \int_{t_0}^{t_f} \int_S \partial_t \mathbf{T}_{glut} : \boldsymbol{\varepsilon}_k \, dV \sin\omega_k t \, dt$$

↑ glut-rate

Moment tensor D&T Sect. 5.4.1

skip this for now
go to page 20

In the limit of long wavelengths ($\gg$ source dimensions) and long periods ($\gg$ source duration) we can approximate

$$\boldsymbol{\varepsilon}_k(x) \begin{Bmatrix} \cos\omega_k t \\ \sin\omega_k t \end{Bmatrix} \approx \boldsymbol{\varepsilon}_k(x_s) \begin{Bmatrix} \cos\omega_k t_s \\ \sin\omega_k t_s \end{Bmatrix}$$

Then x_s, t_s are the ~~[illegible]~~ epicentral location in this point-source approximation

$$a_k = \mathbf{M} : \boldsymbol{\varepsilon}_k(x_s) \cos\omega_k t_s$$
$$b_k = \mathbf{M} : \boldsymbol{\varepsilon}_k(x_s) \sin\omega_k t_s$$

where

$$\mathbf{M} = \int_{t_0}^{t_f} \int_S \partial_t \mathbf{T}_{glut} \, dV \, dt$$

$\mathbf{M} = \mathbf{M}^T$ is the moment tensor — the integrated glut-rate, or equivalently,

$$\mathbf{M} = \int_S \mathbf{T}_{glut,stat} \, dV \,,$$ the integrated final static stress glut

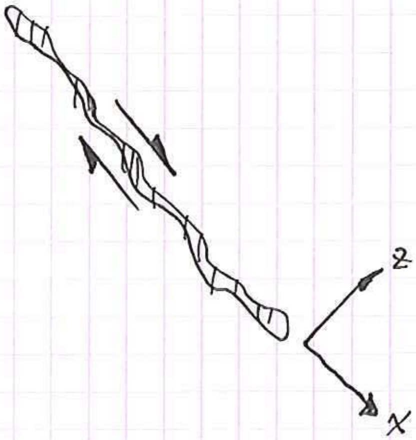
The acceleration response to such a moment tensor source is — using $\cos\omega_k t \cos\omega_k t_s + \sin\omega_k t \sin\omega_k t_s = \cos\omega_k(t-t_s)$:

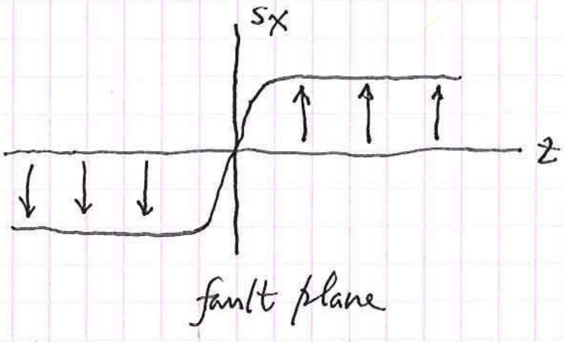
$$a_{\text{[illegible]}}(x,t) = \sum_k \underbrace{\mathbf{M} : \boldsymbol{\varepsilon}_k(x_s)}_{\substack{\text{amplitude} \\ \text{of oscillation}}} \; \underbrace{\mathbf{s}_k(x)}_{\substack{\text{shape} \\ \text{of} \\ \text{oscillation}}} \; \underbrace{\cos\omega_k(t-t_s)}_{\substack{\text{begins} \\ \text{oscillating} \\ \text{at } t=t_s}} \,, \quad t \geq t_f$$

Earthquake fault source   D&T Sect 5.2

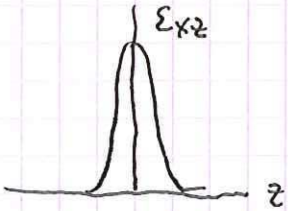
Whence the terminology stress "glut"?

Consider a narrow fault zone such as the San Andreas filled with gouge. The x component of displacement s_x looks like this:

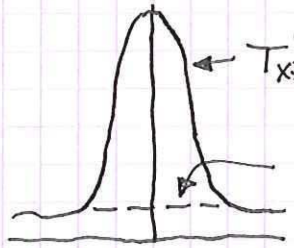




The strain $\varepsilon_{xz} = \frac{1}{2}(\partial_z s_x)$ looks like this:



The Hooke stress and true stress look like this:



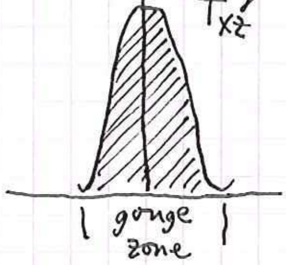
$T_{xz}^{\text{Hooke}} = \mu(\partial_z s_x) + T_{xy}^{\text{initial}}$

$T_{xz}^{\text{true}} \approx T_{xz}^{\text{initial}}$, by continuity

To calculate T_{xz}^{Hooke} we ignore the presence of the gouge and assume that the rigidity $\mu = \mu_{\text{const}} \approx$ constant.

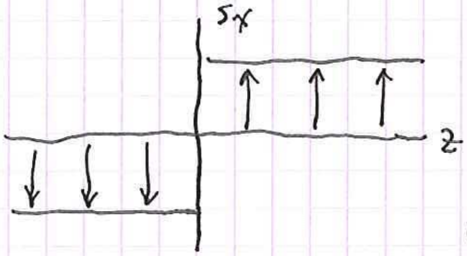
In the competent rock ε_{xz} is very small, of $O(10^{-4})$, whereas in the gouge it is large, $\gg 1$.

$T_{xz}^{\text{glut}} = T_{xz}^{\text{Hooke}} - T_{xz}^{\text{true}}$



There is an excess or glut of Hooke or model stress in the gouge zone, where Hooke's laws fails.

In the limit of a very narrow fault zone:



$$T^{glut}_{xz} = T^{glut}_{zx} = \mu \Delta s \, \delta(z - z_{fault})$$

where $\Delta s = [s_x]^{+}_{-}$ is the total slip on the fault and $\delta(z - z_{fault})$ is a Dirac delta function.

Distribution theory:

To generalize the above simplified analysis it is useful to review some elementary notions from the theory of distributions or generalized functions (L. Schwartz).

A distribution is a continuous linear functional on a space of test functions ϕ. The test functions are assumed to be smooth in $\oplus$ and vanishing on $\partial\oplus$. We denote the scalar that the distribution f assigns to the test function ϕ by:

$$\langle f, \phi \rangle \rightarrow \text{real scalar}$$

$\uparrow$ continuous in this slot

Every ordinary function in $\oplus$ can be regarded as a distribution provided we define

$$\langle f, \phi \rangle = \int_{\oplus} f \phi \, dV$$

when it is important to distinguish functions and distributions we write the distribution associated with a function f in the form $\mathcal{D}f$.

By analogy we also frequently write

$$\langle f, \phi \rangle = \int_{\Phi} f \phi \, dV$$

for a more general distribution, the "integration" being purely symbolic. All distributions of the form $\mathcal{D}f$ are said to be regular; all others are singular.

If f is an ordinary differentiable function (regular distribution) we can write

$$\int_{\Phi} (\nabla f) \phi \, dV = - \int_{\Phi} f \nabla \phi \, dV \qquad \text{(since the } \phi\text{'s vanish on } \partial \Phi\text{)}$$

More generally we define the gradient of a singular distribution f by:

$$\langle \nabla f, \phi \rangle \equiv - \langle f, \nabla \phi \rangle$$

If the test functions ϕ are smooth enough then every distribution can be differentiated any number of times in this sense.

The most familiar example of a singular distribution is the Dirac delta distribution (or "function") defined by

$$\langle \delta_\xi, \phi \rangle = \int_{\oplus} \delta_\xi \phi \, dV = \phi(\xi)$$ ← no ordinary function has this property — just selects $\phi(\xi)$ at a point ξ.

in book $\langle \delta_0, \phi \rangle = \phi(x_0)$

A more common notation for this:

$$\int_{\oplus} \delta(x-\xi)\, \phi(x)\, d^3x = \phi(\xi)$$

The gradient $\nabla \delta_\xi$ is defined in accordance with the general definition by

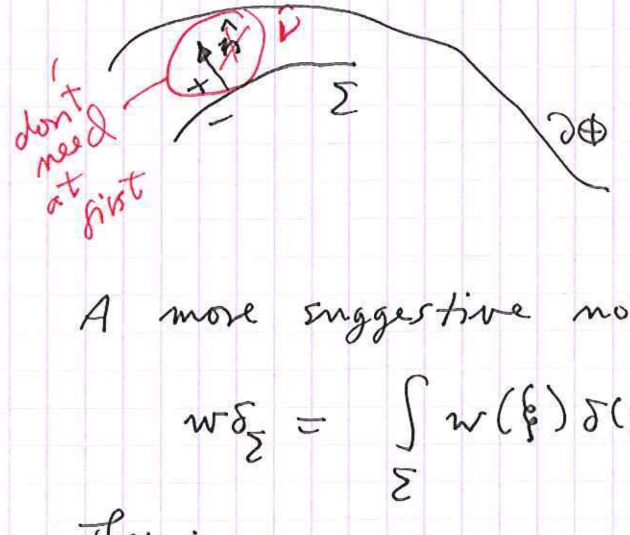
$$\langle \nabla \delta_\xi, \phi \rangle = \int_{\oplus} \nabla_x \delta(x-\xi)\, \phi(x)\, d^3x$$

$$= - \int_{\oplus} \delta(x-\xi)\, \nabla_x \phi(x)\, d^3x = - \langle \delta_\xi, \nabla\phi \rangle.$$

$$= - \nabla\phi(\xi)$$

A useful singular distribution in the present context is defined as follows:

Let Σ be a smooth 2-d surface in $\oplus$ and let w a smooth ordinary function on Σ.



Define the distribution $w\delta_\Sigma$ by

$$\langle w\delta_\Sigma, \phi \rangle = \int_{\Sigma} w\phi \, d\Sigma$$

A more suggestive notation:

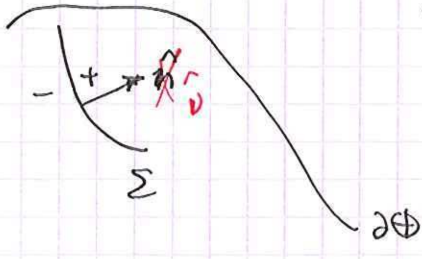
$$w\delta_\Sigma = \int_{\Sigma} w(\xi)\, \delta(x-\xi)\, d^2\xi$$

Then:

$$\langle w\delta_\Sigma, \phi \rangle = \int_{\oplus} \int_{\Sigma} w(\xi)\, \delta(x-\xi)\, \phi(x)\, d^2\xi\, d^3x = \int_{\Sigma} w(\xi)\phi(\xi)\, d^2\xi$$

We can regard $w\delta_\Sigma$ as a weighted "distribution" of Dirac deltas on Σ just as $\sum_k w_k \delta(x-\xi_k)$ is a weighted discrete "distribution".
The "value" of $w\delta_\Sigma$ (this concept can be made rigorous) is zero everywhere except on Σ just as the "value" of $\sum_k w_k \delta(x-\xi_k)$ is zero everywhere except at the points ξ_k.

We now make a calculation — suppose that f is an ordinary function that is smooth everywhere in Φ except on a surface Σ where it exhibits a jump discontinuity $[f]^\pm$



What is ∇f? It does not exist as an ordinary function everywhere in Φ, notably on Σ. But we can calculate $\nabla(\mathcal{D}f)$.

$$\langle \nabla(\mathcal{D}f), \phi \rangle \equiv -\langle \mathcal{D}f, \nabla\phi \rangle$$

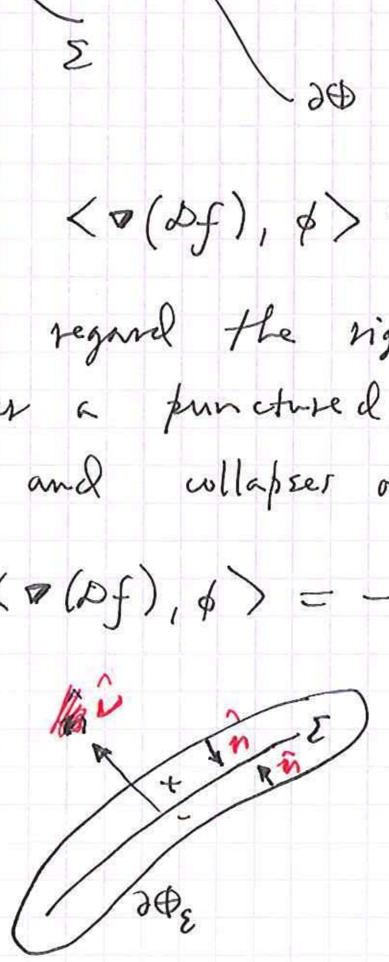
We regard the right side as an integral over a punctured volume that envelops Σ and collapses onto it in the limit $\varepsilon \to 0$

$$\langle \nabla(\mathcal{D}f), \phi \rangle = -\lim_{\varepsilon\to 0} \int_{\Phi-\Phi_\varepsilon} f \, \nabla\phi \, dV$$

now an ordinary smooth function in this domain

why do we do this — because we want to be able to integrate by parts

better to call this $\hat{\nu}$



We can apply Gauss' theorem <u>before</u> taking the limit:

$$-\int_{\oplus-\oplus_\varepsilon} f \, \nabla\phi \, dV = -\int_{\partial\oplus_\varepsilon} \hat{n} f \phi \, d\Sigma + \int_{\oplus-\oplus_\varepsilon} \nabla f \, \phi \, dV$$

unit <u>inward</u> normal to $\partial\oplus_\varepsilon$

Now take limit $\varepsilon \to 0$

$$= \int_{\Sigma} \hat{n} \, [f]^{\pm} \, \phi \, d\Sigma + \int_{\oplus} \nabla f \, \phi \, dV$$

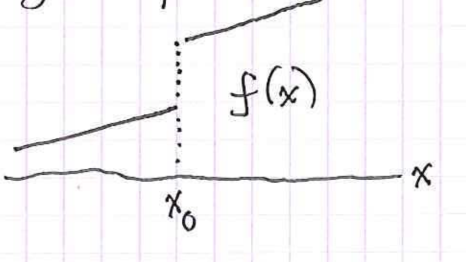
In conclusion:

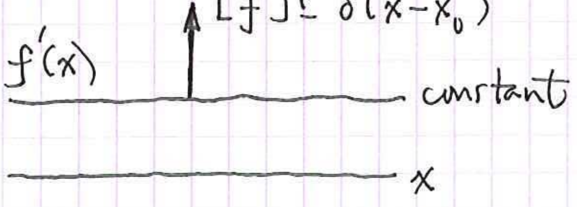
$$\langle \nabla(\mathcal{D}f), \phi \rangle = \langle \mathcal{D}(\nabla f) + \overset{\hat{\nu}}{\hat{n}} [f]^{\pm} \delta_\Sigma , \phi \rangle$$

Since these two distributions assign the same scalar to every test function ϕ, they must be equal:

$$\nabla(\mathcal{D}f) = \mathcal{D}(\nabla f) + \overset{\hat{\nu}}{\hat{n}} [f]^{\pm} \delta_\Sigma$$

Loosely speaking, $\nabla(\mathcal{D}f)$ consists of the "ordinary" gradient ∇f, which is well defined everywhere except on Σ, plus a delta function contribution on Σ arising from the discontinuity $[f]^{\pm}$. Generalization of differentiation in 1-d.





<u>Ideal</u> <u>fault</u> : a surface Σ in $\oplus$ across which there is a <u>tangential</u> slip discontinuity $\Delta s = [s]^{\pm}$ satisfying $\underset{\hat{\nu}}{\hat{n}} \cdot \Delta s = 0$ (no opening up or interpenetration).

The Hooke stress $\mathbb{T}_{\text{Hooke}} = \mathbf{C} : \boldsymbol{\varepsilon} = \mathbf{C} : \nabla s$ does not exist everywhere, notably on Σ, as an ordinary function so we consider the associated distribution:

$$\mathbb{T}_{\text{Hooke}} = \mathbf{C} : \nabla(\mathcal{D} s)$$

$$T_{ij}^{\text{Hooke}} = C_{ijkl}\, \partial_k (\mathcal{D} s_l)$$ — a singular distribution

The true physical stress is, on the other hand, well defined everywhere. The associated distribution is regular:

discontinuous but non-singular

$$\mathcal{D}(\mathbb{T}_{\text{true}}) = \mathbf{C} : \mathcal{D}(\nabla s)$$

$$\mathcal{D}(T_{ij}^{\text{true}}) = C_{ijkl}\, \mathcal{D}(\partial_k s_l)$$

The stress glut is defined as a singular distribution by

$$\mathbb{T}_{\text{glut}} = \text{[illegible]}\ \mathbb{T}_{\text{Hooke}} - \mathcal{D}(\mathbb{T}_{\text{true}})$$

$$= \mathbf{C} : \left[\nabla(\mathcal{D} s) - \mathcal{D}(\nabla s) \right]$$

$$= \mathbf{C} : \underset{\nu}{\cancel{\hat{n}}}\, \Delta s\, \delta_\Sigma$$

lousy name — wrong units

Define the ~~stress glut density~~ on Σ:

$$\mathbf{m} = \mathbf{C} : \underset{\nu}{\cancel{\hat{n}}}\, \Delta s$$

$$m_{ij} = C_{ijkl}\, \overset{\nu_k}{\cancel{n_k}}\, \Delta s_l$$

units of $\mathbf{m}$: $\dfrac{\text{force} \times \text{distance}}{\text{area}} = \dfrac{ml/t^2 \times l}{l^2} = \dfrac{m}{t^2}$

Then $\mathbb{T}_{\text{glut}} = \mathbf{m}\, \delta_\Sigma$ — confined to fault surface as expected

end here Thurs Mar 2

Quantity $\mathbf{m}$ also called moment tensor density.

The product of a discontinuous function times a Dirac delta is not defined so we require that

Ampuero & Dahlen BSSA (2005)

$$[\mathbf{C}]^{+}_{-} = 0$$ — elastic parameters continuous across the fault Σ.

The equivalent body and surface force densities for an ideal fault are

$$\mathbf{f} = -\mathbf{m} \cdot \nabla \delta_{\Sigma}$$

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \mathbf{m})\, \delta_{\Sigma}$$

$\hat{\mathbf{n}}$: normal to $\partial\oplus$, not Σ

Note that this is completely general, for dynamic, time-dependent faulting in an anisotropic Earth.

skip

The moment tensor of an ideal fault is

$$\mathbf{M} = \int_{S} \mathbf{T}_{\text{glut, stat}}\, dV = \int_{\oplus} \mathbf{T}_{\text{glut, stet}}\, dV$$

$$= \int_{\oplus} \mathbf{m}\, \delta_{\Sigma}\, dV$$

$$\mathbf{M} = \int_{\Sigma} \underset{\text{moment tensor / unit area}}{\mathbf{m}}\, d\Sigma$$

If the Earth is isotropic:

$$C_{ijkl} = \underbrace{\left(\kappa - \tfrac{2}{3}\mu\right)\delta_{ij}\,\delta_{kl}}_{\text{no contribution since } n_k \Delta s_k = 0} + \mu(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$$

$$\mathbf{m} = \mu \Delta s\,(\underset{\hat{\nu}}{\hat{\mathbf{n}}}\hat{\mathbf{e}} + \hat{\mathbf{e}}\underset{\hat{\nu}}{\hat{\mathbf{n}}}) \quad \text{where} \quad \Delta \mathbf{s} = \Delta s\, \underset{\text{slip direction on fault}}{\hat{\mathbf{e}}}$$

skip

The moment tensor in this case is

$$\mathbf{M} = \int_{\Sigma} \mu \Delta s\,(\underset{\hat{\nu}}{\hat{\mathbf{n}}}\hat{\mathbf{e}} + \hat{\mathbf{e}}\underset{\hat{\nu}}{\hat{\mathbf{n}}})\, d\Sigma$$

If the fault [illegible] surface is planar & the slip is uni-directional ($\underset{\hat\nu}{\hat{n}}$ & $\hat{e}$ constant)

$$\mathbf{M} = M_0\,(\underset{\hat\nu}{\hat{n}}\hat{e} + \hat{e}\underset{\hat\nu}{\hat{n}})$$

where $M_0 = \int_\Sigma \mu\,\Delta s\,d\Sigma$, the scalar seismic moment first defined by Aki

slip

Note the fault plane – auxiliary plane ambiguity: cannot distinguish $\overset{\hat\nu}{\hat{n}}$ from $\hat{e}$

For the largest events $M_0 \sim 10^{30}$ dyne-cm (10^{23} N-m)

e.g. 1964 Alaskan 'quake

$\mu \sim 3 \times 10^{11}$ dyne/cm^2

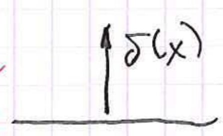
$\overline{\Delta s} = 10$ m

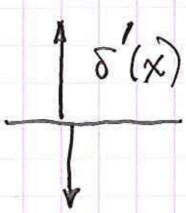
$A = 1000$ km × 250 km

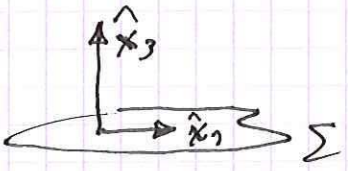
$M_0 \sim 7.5 \times 10^{29}$ dyne-cm (Kanamori)

The equivalent point-source body force is a classical double couple: $\mathbf{f} = -\mathbf{M}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_s)$

do this for moment density







$\hat{n} = \hat{x}_3$

$\hat{e} = \hat{x}_1$

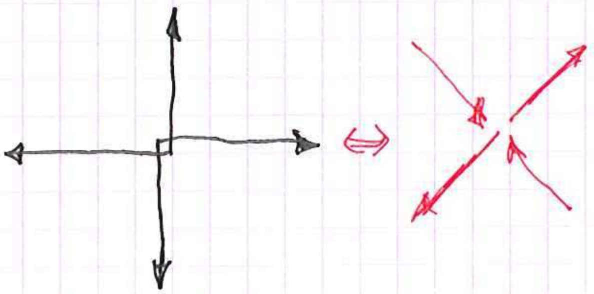
Then $M_{13} = M_{31} = M_0$

All other $M_{ij} = 0$

$$f_1 = -M_0\,\delta(x_1)\,\delta(x_2)\,\delta'(x_3)$$

$$f_2 = -M_0\,\delta'(x_1)\,\delta(x_2)\,\delta(x_3)$$

Note – obviously no net force or torque



More generally f is a "distribution" of double couples over the fault surface Σ — this is an exact dynamical result for all frequencies & wavelengths.

Harvard CMT project : solves for $\mathbf{M}$ as well as an updated location $x_s + \Delta x$, $t_s + \Delta t$. In this case the acceleration response is

$$a(x,t) = \sum_k \mathbf{M} : \varepsilon_k(x_s)\, s_k(x) \cos \omega_k (t - t_s)$$

$$+ \sum_k \Delta x\, \mathbf{M} : \nabla\varepsilon_k(x_s)\, s_k(x) \cos \omega_k (t - t_s)$$

$$+ \sum_k \omega_k \Delta t : \varepsilon_k(x_s)\, s_k(x) \sin \omega_k (t - t_s)$$

Linear inverse problem for $\mathbf{M}$, Δx, Δt
Can be shown that the centroid shift ~~[illegible]~~ is given by:

$$\Delta x = \frac{1}{M_0} \int_\Sigma (x - x_s) \mu \Delta s_{stat}\, d\Sigma$$

$$\Delta t = \frac{1}{M_0} \int_{t_0}^{t_f} \int_\Sigma (t - t_s) \mu \partial_t \Delta s\, d\Sigma\, dt$$

Centroid of source in space-time: may differ from rupture initiation point.

To compare with near-field geodetic observations and for other reasons often seek best-fitting double couple source. The constraints

$$\operatorname{trace} \mathbf{M} = 0$$

$$\det \mathbf{M}$$

guarantee this.

The first is linear & easily imposed – the second is not. Customary to impose trace $\mathbf{M} = 0$ and find the best-fitting <u>deviatoric</u> tensor $\mathbf{M}'$ by fitting seismograms. Then find best-fitting $\mathbf{M}_{dc}$ by

$$(\mathbf{M}_{dc} - \mathbf{M}') : (\mathbf{M}_{dc} - \mathbf{M}') = \text{minimum}$$

Solution found by <u>diagonalization</u> of $\mathbf{M}'$:

$$\mathbf{M}' = \begin{pmatrix} M_{maj} & & \\ & -M_{maj} - M_{min} & \\ & & M_{min} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} M_{maj} & & \\ & -M_{maj} & \\ & & 0 \end{pmatrix}}_{\substack{\text{major double}\\ \text{couple}\\ \text{(best-fitting)}}} + \underbrace{\begin{pmatrix} 0 & & \\ & -M_{min} & \\ & & M_{min} \end{pmatrix}}_{\substack{\text{minor double}\\ \text{couple}}}$$

where $|M_{maj}| \geq |M_{min}|$

A measure of the amount by which $\mathbf{M}'$ deviates from a double couple is given by

$$\varepsilon = \frac{M_{max} + M_{min}}{\max(|M_{max}|, |M_{min}|)}$$

Then $\varepsilon = 0$ corresponds to a double couple and $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$ in general.

In the Harvard catalogue, 4% of the mechanisms are significantly non-double-couple $|\varepsilon| \geq \frac{1}{3}$. The most likely cause is <u>curvature</u> of the <u>fault plane</u>. Ekstrom (1994) shows that several can be associated with <u>volcanic ring faults</u>.

Anelasticity and Attenuation : (D&T ch. 6)

Hooke's law specifies that the stress $\mathbf{T}$ at $\mathbf{x}, t$ depends only upon the instantaneous $\nabla \mathbf{s}$ at the same point $\mathbf{x}$ and time t. In an isotropic material:

$$\mathbf{T} = \kappa(\nabla\cdot\mathbf{s})\mathbf{I} + 2\mu\,\boldsymbol{\delta}$$

how to generalize to anisotropic — what are the complementary $\sigma(t)$ and $\varepsilon(t)$

To account for damping we generalize this, retaining the linearity & spatial locality :

$$\mathbf{T}(x,t) = \int_{-\infty}^{t} \kappa(t-t')\,\nabla\cdot\partial_{t'}\mathbf{s}(x,t')\,\mathbf{I}\,dt' + \int_{-\infty}^{t} 2\mu(t-t')\,\partial_{t'}\boldsymbol{\delta}(x,t')\,dt'$$

$\mathbf{T}$ at time t now depends on entire previous strain history — Boltzmann superposition principle. For simplicity lets denote strain by scalar $\varepsilon(t)$ and complementary stress by $\sigma(t)$:

modulus $M(t)$	$\varepsilon(t)$	$\sigma(t)$	remark
$\kappa(t)$	$\mathrm{tr}\,\boldsymbol{\varepsilon} = \nabla\cdot\mathbf{s}$	$\frac{1}{3}\mathrm{tr}\,\mathbf{T} = -\delta p$	isotropic
$\mu(t)$	$\boldsymbol{\delta}$	$\mathbf{T} - \frac{1}{3}(\mathrm{tr}\,\mathbf{T})\mathbf{I}$	deviatoric

Then write

$$\sigma(t) = \int_{-\infty}^{t} M(t-t')\,\partial_{t'}\varepsilon(t')\,dt'$$

lower limit $-\infty$: entire past history

upper limit t : causality, no dependence on future

Can alternatively relate strain to stress by

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-t')\,\partial_{t'}\sigma(t')\,dt'$$

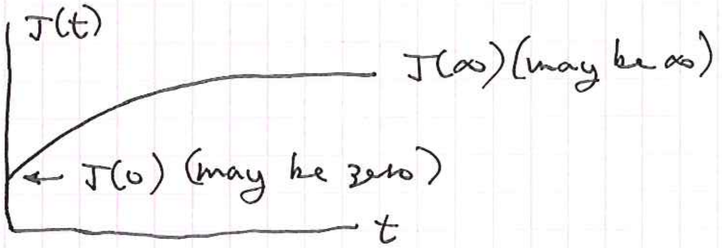
$M(t)$ called stress relaxation function.

$J(t)$ called creep function.

In a laboratory creep experiment — apply a unit step stress $\sigma(t) = H(t)$. Then

$$\varepsilon(t) = J(t)$$

$J(t)$ is the strain response to a unit step stress

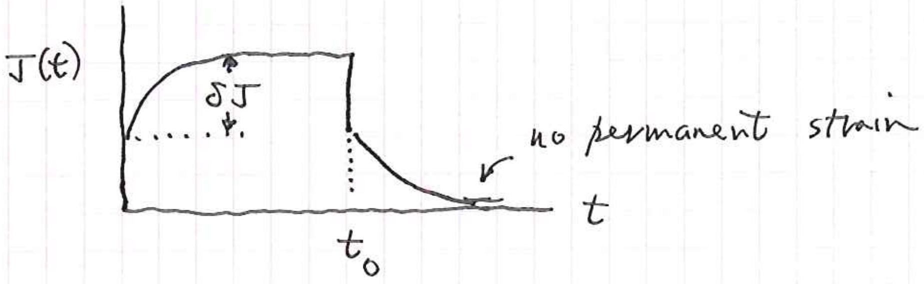


Define: $J_u = J(0)$ — unrelaxed or instantaneous elastic compliance

$J_r = J(\infty)$ — relaxed compliance ($= \infty$ in materials that exhibit steady-state creep)

An anelastic material has $J_u > 0$ and $J_r < \infty$ — such a material exhibits complete strain recovery in a creep cycle

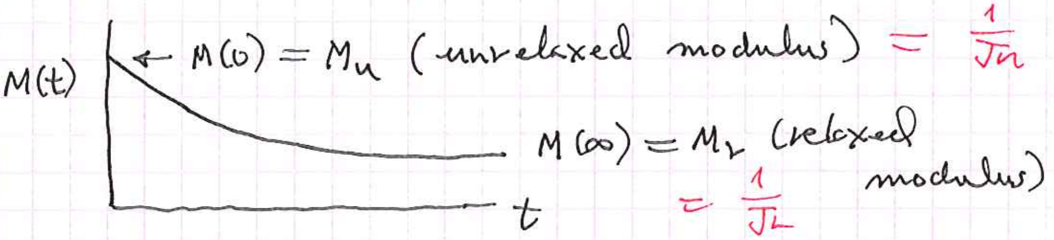
$$\sigma(t) = H(t) - H(t-t_0)$$



This a reasonable model on seismic time scales — no good for isostasy & post-glacial rebound.

The stress relaxation function is likewise the stress response to a unit step strain

$$\varepsilon(t) = H(t) \iff \sigma(t) = M(t)$$

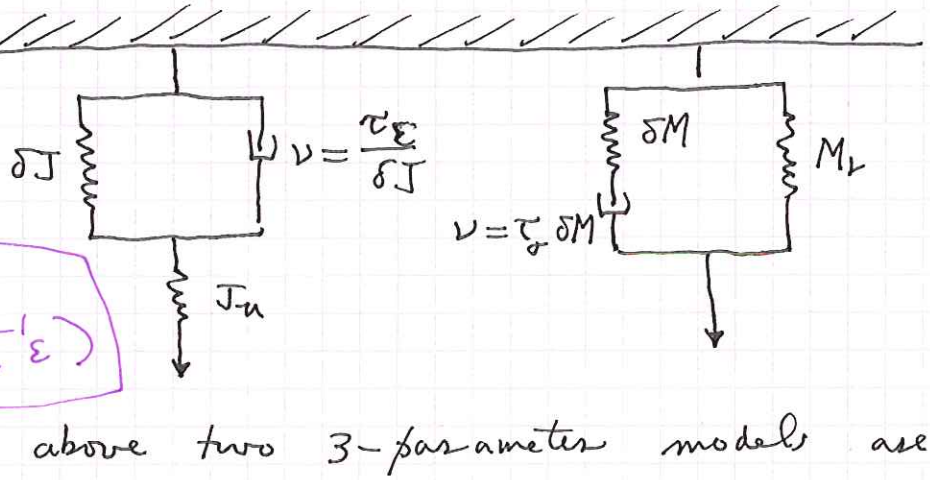


The stress required to maintain the unit strain relaxes with time — this the reason for the nomenclature. The difference $\delta M = M_u - M_r$ called the modulus defect.

end here lecture 11 with discussion of Maxwell & Kelvin Voigt

Standard linear solid:

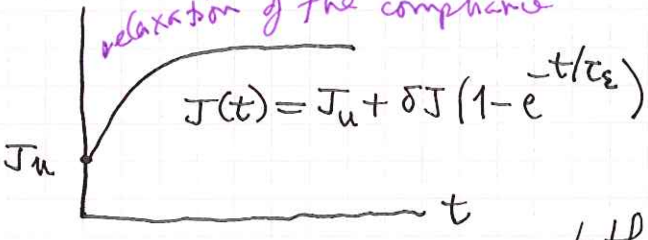
Simplest example of a material with both an instantaneous elastic response and complete creep recovery.

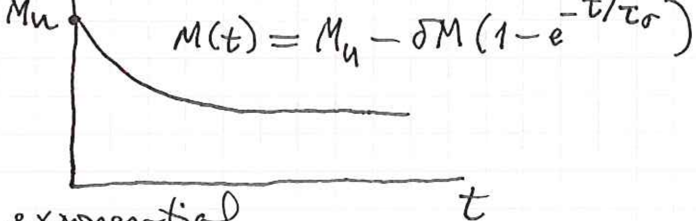


$$\boxed{\begin{gathered}\dot\sigma + \tau_\sigma^{-1}\sigma \\ = M_u(\dot\varepsilon + \tau_\varepsilon^{-1}\varepsilon)\end{gathered}}$$

The above two 3-parameter models are equivalent provided $\tau_\sigma = \tau_\varepsilon (J_u/J_r)$:

relaxation of the modulus or modulus defect





both exponential

Harmonic variations:

systematically use ω for real freq and $\nu = \omega + i\alpha$ for complex

We have been considering static loading.

Suppose now that

$$\sigma(t) = \mathrm{Re}\,[\sigma(\nu) e^{i\nu t}] \qquad \text{where} \quad \mathrm{Im}\,\nu \le 0$$

$$\varepsilon(t) = \mathrm{Re}\,[\varepsilon(\nu) e^{i\nu t}]$$

How are the complex amplitudes $\sigma(\nu)$ and $\varepsilon(\nu)$ related?

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-t')\, \partial_{t'} \sigma(t')\, dt'$$

let $\xi = t - t'$; $d\xi = -dt'$; $\partial_{t'} = -\partial_\xi$

$$= -\int_0^\infty J(\xi)\, \partial_\xi \sigma(t-\xi)\, d\xi$$

$$= -\int_0^\infty J(\xi)\, \partial_\xi\, \mathrm{Re}\,[\sigma(\nu) e^{i\nu(t-\xi)}]\, d\xi$$

$$= \mathrm{Re}\,\left[\, i\nu \int_0^\infty J(\xi) e^{-i\nu\xi}\, d\xi \,\right] \sigma(\nu) e^{i\nu t}$$

In summary:

$$\varepsilon(\nu) = J(\nu)\sigma(\nu)$$

$$\sigma(\nu) = M(\nu)\,\varepsilon(\nu)$$

$$M(\nu)\,J(\nu) = 1$$

$$J(\nu) = i\nu \int_0^\infty J(t) e^{-i\nu t}\, dt \qquad \text{complex compliance}$$

$$M(\nu) = i\nu \int_0^\infty M(t) e^{-i\nu t}\, dt \qquad \text{complex modulus}$$

Both are well-defined and analytic in the lower half-plane $\mathrm{Im}\,\nu \le 0$. Analytic continuation into upper half-plane will be singular. Note that $J(-\nu^*) = J^*(\nu)$ and $M(-\nu^*) = M^*(\nu)$ in lower halfplane

Low and high-frequency limits:

$$\lim_{\nu\to 0} M(\nu) = \lim_{\nu\to 0} 1/J(\nu) = M_r = 1/J_r$$ relaxed at low freq.

$$\lim_{\nu\to\infty} M(\nu) = \lim_{\nu\to\infty} 1/J(\nu) = M_u = 1/J_r$$ unrelaxed at high freq.

On real frequency axis conventional to define real and imaginary parts by

$$M(\omega) = M_1(\omega) + i M_2(\omega)$$

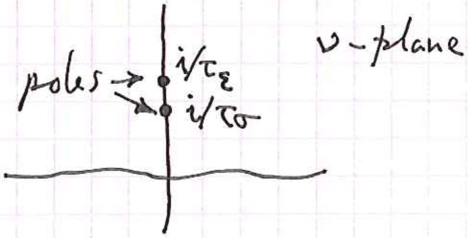
$$J(\omega) = J_1(\omega) - i J_2(\omega)$$

Then $M_1(\omega)$ and $J_1(\omega)$ are <u>even</u> and $M_2(\omega)$ and $J_2(\omega)$ are <u>odd</u> functions of real ω.

The <u>standard linear solid</u> has:

$$M(\omega) = M_u - \delta M\,(1 + i\nu\tau_\sigma)^{-1}$$

$$J(\nu) = J_u + \delta J\,(1 + i\nu\tau_\varepsilon)^{-1}$$

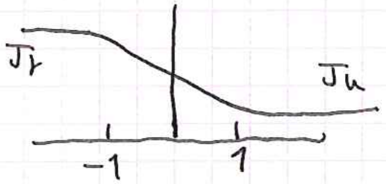


On the real axis:

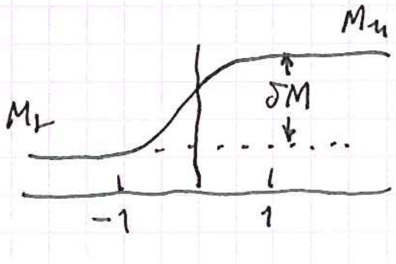
$$M_1(\omega) = M_u - \frac{\delta M}{1+\omega^2\tau_\sigma^2} \qquad J_1(\omega) = J_u + \frac{\delta J}{1+\omega^2\tau_\varepsilon^2}$$

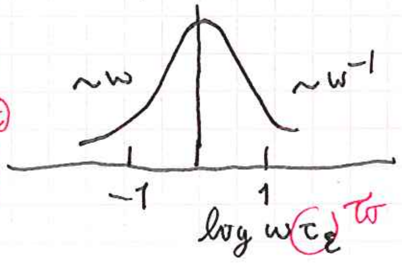
$$M_2(\omega) = \delta M\,\frac{\omega\tau_\sigma}{1+\omega^2\tau_\sigma^2} \qquad J_2(\omega) = \delta J\,\frac{\omega\tau_\varepsilon}{1+\omega^2\tau_\varepsilon^2}$$

Plotted versus log ω:



normal dispersion





← called Debye relaxation peaks →

x-axis labels wrong in D&T Fig. 6.5

Energy dissipation and Q(ω):

The energy dissipated against internal friction per unit volume in a full cycle of (forced) oscillation at real frequency ω is

$$\oint \partial_t E \, dt = \oint \sigma \partial_t \varepsilon \, dt$$

$$\varepsilon(t) = \mathrm{Re}\left[\varepsilon(\omega) e^{i\omega t}\right]$$

$$\sigma(t) = \mathrm{Re}\left[\sigma(\omega) e^{i\omega t}\right] \qquad \sigma(\omega) = M(\omega)\varepsilon(\omega)$$

write $\varepsilon(\omega) = |\varepsilon(\omega)| e^{i\Phi(\omega)}$

$M(\omega) = |M(\omega)| e^{i\phi(\omega)}$

Then $\partial_t \varepsilon = -\omega |\varepsilon| \sin(\omega t + \Phi)$

$\sigma = |M||\varepsilon| \cos(\omega t + \Phi + \phi)$

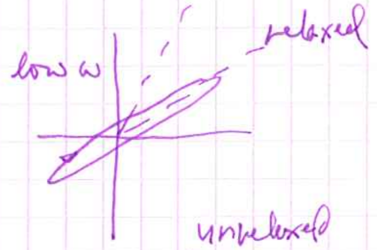
$$\oint \sigma \partial_t \varepsilon \, dt = -\omega |M||\varepsilon^2| \underbrace{\oint \cos(\omega t + \Phi + \phi) \sin(\omega t + \Phi) \, dt}_{*}$$

$$* = \oint \cos(\omega t + \Phi) \sin(\omega t + \Phi) \, dt \cos\phi$$

$$- \oint \sin^2(\omega t + \Phi) \, dt \sin\phi$$

$$= -\frac{1}{2} \frac{2\pi}{\omega} \sin\phi = -\frac{\pi}{\omega} \sin\phi$$

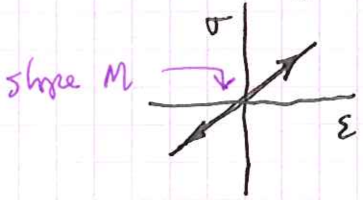
average ↗ ↑ length of cycle



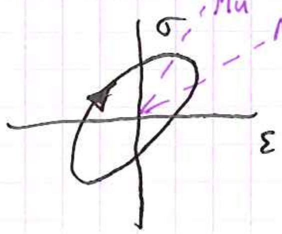
$$\oint \partial_t E \, dt = \pi |M| \sin\phi \, |\varepsilon|^2 = \pi M_2 |\varepsilon|^2$$

$$\oint \partial_t E \, dt = \pi M_2(\omega) |\varepsilon(\omega)|^2 = \pi J_2(\omega) |\sigma(\omega)|^2$$

Imaginary part of modulus is thus a measure of the rate of anelastic damping:



elastic



anelastic (hysteresis loop)

The intrinsic quality factor is defined by

$$Q(\omega) = \frac{M_1(\omega)}{M_2(\omega)} = \frac{J_1(\omega)}{J_2(\omega)}$$

We also have $Q^{-1}(\omega) = \tan\phi(\omega)$ where $\phi(\omega)$ is the phase lag of the stress behind the strain in harmonic loading.

The average elastic energy stored in the springs can be shown to be (for a special class of materials that can be represented by a network of springs & dashpots)

$$\langle E \rangle = \frac{1}{4} M_1(\omega)\,|\varepsilon(\omega)|^2 = \frac{1}{4}|J_1(\omega)|\,|\sigma(\omega)|^2$$

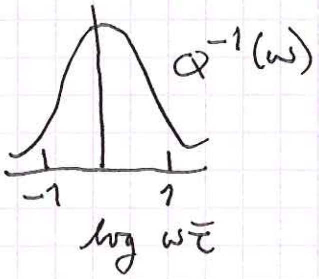
In this case $Q(\omega)$ has an energetic interpretation:

$$\frac{4\pi}{Q} = \frac{1}{\langle E \rangle}\oint \partial_t E\,dt$$ = fractional energy dissipated — converted to heat — per cycle

Example: standard linear solid

$$Q^{-1}(\omega) = \frac{\delta M}{\sqrt{M_u M_r}}\left(\frac{\omega\bar{\tau}}{1+\omega^2\bar{\tau}^2}\right) \quad \text{where } \bar{\tau} = \sqrt{\tau_\sigma \tau_\varepsilon}$$

Another Debye peak:
The dissipation in this case limited to a narrow frequency band.



Anelasticity in the Earth (and in geological materials in general) seems on the other hand to be characterized by constant (frequency-independent) $Q(\omega)$.

Relaxation spectrum:

$Q(\omega)$ in the Earth is observed to be roughly *independent* of frequency within the band of interest in global seismology — from 0.3 mHz to 1 Hz. A useful model of such a material can be constructed by superposing standard linear solids with a continuous spectrum of relaxation times τ:

$$M(t) = M_u - \int_0^\infty \tau^{-1} \Upsilon(\tau) \left[1 - e^{-t/\tau} \right] d\tau$$

$$M(\nu) = M_u - \int_0^\infty \tau^{-1} \Upsilon(\tau) \, (1 + i\nu\tau)^{-1} \, d\tau$$

For real frequencies:

$$M_1(\omega) = M_u - \int_0^\infty \frac{\Upsilon(\tau)}{1+\omega^2\tau^2} \frac{d\tau}{\tau}$$

$$M_2(\omega) = \int_0^\infty \Upsilon(\tau) \frac{\omega\tau}{1+\omega^2\tau^2} \frac{d\tau}{\tau}$$

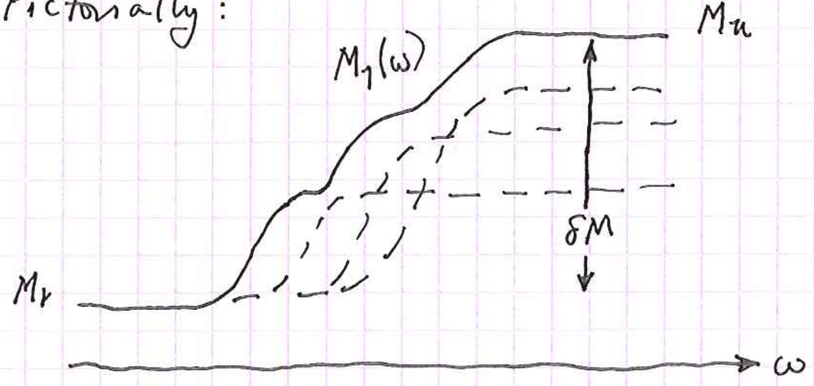
The weight factor $\Upsilon(\tau)$ is called the *relaxation spectrum* — it is a measure of the total contribution to the modulus defect from anelastic processes in the material with relaxation times between τ and $\tau + d\tau$:

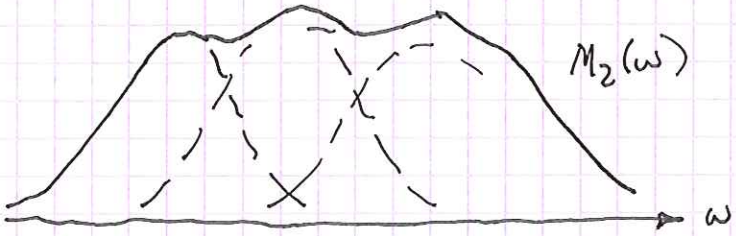
← this extra factor to force a logarithmic scaling is conventional

$$\delta M = \int_0^\infty \tau^{-1} \Upsilon(\tau) \, d\tau$$

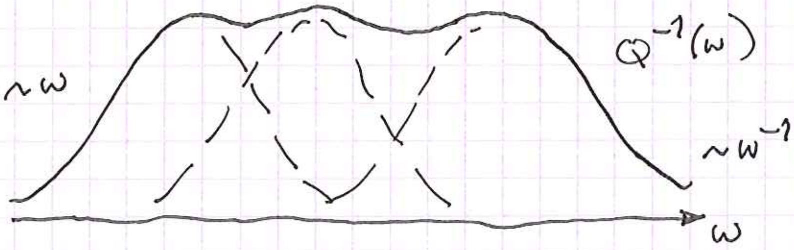
$$= \int_0^\infty \Upsilon(\tau) \, d(\ln\tau)$$

Pictorially:





The region where $M_2(\omega)$ is significant is called the absorption band — no significant attenuation above or below. $Q(\omega) = M_1(\omega)/M_2(\omega)$ looks ~ the same:



By adjusting the spectrum $Y(\tau)$ can attain any desired variation of $Q(\omega)$ within the absorption band.

Approximate relations:

Suppose now that $Y(\tau)$ is slowly varying over a broad range of relaxation times τ. The Debye peak $\frac{\omega\tau}{1+\omega^2\tau^2}$ is peaked at $\omega\tau = 1$ and neglible outside $\frac{1}{10} \leq \omega\tau \leq 10$.

Can write, approximately,

$$M_2(\omega) \approx \omega Y(\tfrac{1}{\omega}) \int_0^\infty \frac{d\tau}{1+\omega^2\tau^2}$$

$$= Y(\tfrac{1}{\omega}) \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} Y(\tfrac{1}{\omega})$$

see D&T eq. (6.72)

The function $\frac{1}{1+\omega^2\tau^2}$ has a steplike character over the same range $\frac{1}{10} \leq \omega\tau \leq 10$.

Approximate by an abrupt step:

$$M_1(\omega) \approx M_u - \int_0^{1/\omega} \tau^{-1} Y(\tau)\, d\tau$$

Differentiate:

$$\frac{dM_1(\omega)}{d\omega} \approx \frac{1}{\omega} Y(\tfrac{1}{\omega})$$

Comparing we obtain:

$$\frac{dM_1(\omega)}{d\omega} \approx \frac{2M_2(\omega)}{\pi\omega} \quad , \quad \text{or}$$

$$\frac{d \ln M_1(\omega)}{d \ln \omega} \approx \frac{2}{\pi Q(\omega)}$$

Upon integrating:

$$\frac{M_1(\omega)}{M_1(\omega_0)} \approx \exp\left[\frac{2}{\pi}\int_{\omega_0}^{\omega} \frac{d\omega'}{\omega' Q(\omega')}\right] \approx 1 + \frac{2}{\pi}\int_{\omega_0}^{\omega} \frac{d\omega'}{\omega' Q(\omega')}$$

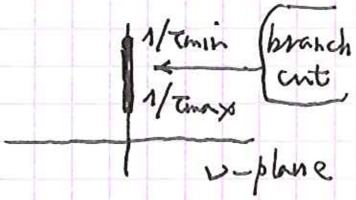
where the last approximation assumes that $Q(\omega) \gg 1$.

Similar manipulations lead to a number of other approximations, e.g. $M_1(\omega)\, J_1(\omega) \approx 1$ and $M(t)\, J(t) \approx 1$.

Constant-Q absorption band model (Liu, Anderson & Kanamori 1976):

Consider the particular form of the spectrum

$$Y(\tau) = \begin{cases} \dfrac{\delta M}{\ln(\tau_{max}/\tau_{min})} & \text{if } \tau_{min} \le \tau \le \tau_{max} \\ 0 & \text{otherwise} \end{cases}$$



Evaluating the integral we find

$$M(\nu) = M_u - \frac{1}{2}\left[\frac{\delta M}{\ln(\tau_{max}/\tau_{min})}\right] \ln\left[\frac{i\nu + 1/\tau_{min}}{i\nu + 1/\tau_{max}}\right]$$

For real frequencies:

$$M_1(\omega) = M_u - \frac{1}{2}\left[\frac{\delta M}{\ln(\tau_{max}/\tau_{min})}\right] \ln\left[\frac{1+\omega^2\tau_{min}^2}{\tau_{min}^2/\tau_{max}^2 + \omega^2\tau_{min}^2}\right]$$

$$M_2(\omega) = \left[\frac{\delta M}{\ln(\tau_{max}/\tau_{min})}\right] \arctan\left[\frac{\omega(\tau_{max}-\tau_{min})}{1+\omega^2\tau_{min}\tau_{max}}\right]$$

The quality factor $Q(\omega) = \dfrac{M_1(\omega)}{M_2(\omega)}$ is independent of frequency in the band $1/\tau_{max} \ll \omega \ll 1/\tau_{min}$.

It's value there is

$$Q^{-1} = \frac{\pi\,\delta M}{2M_u \ln(\tau_{max}/\tau_{min})}$$

The real modulus within the same band is

$$M_1(\omega) \approx M_u\left[1 + \frac{2}{\pi Q}\ln(\omega\tau_{min})\right]$$

The ratio of the modulus at two frequencies is:

$$\frac{M_1(\omega)}{M_1(\omega_0)} \approx 1 + \frac{2}{\pi Q}\ln\left(\frac{\omega}{\omega_0}\right)$$ — this can also be obtained from our earlier approximation.

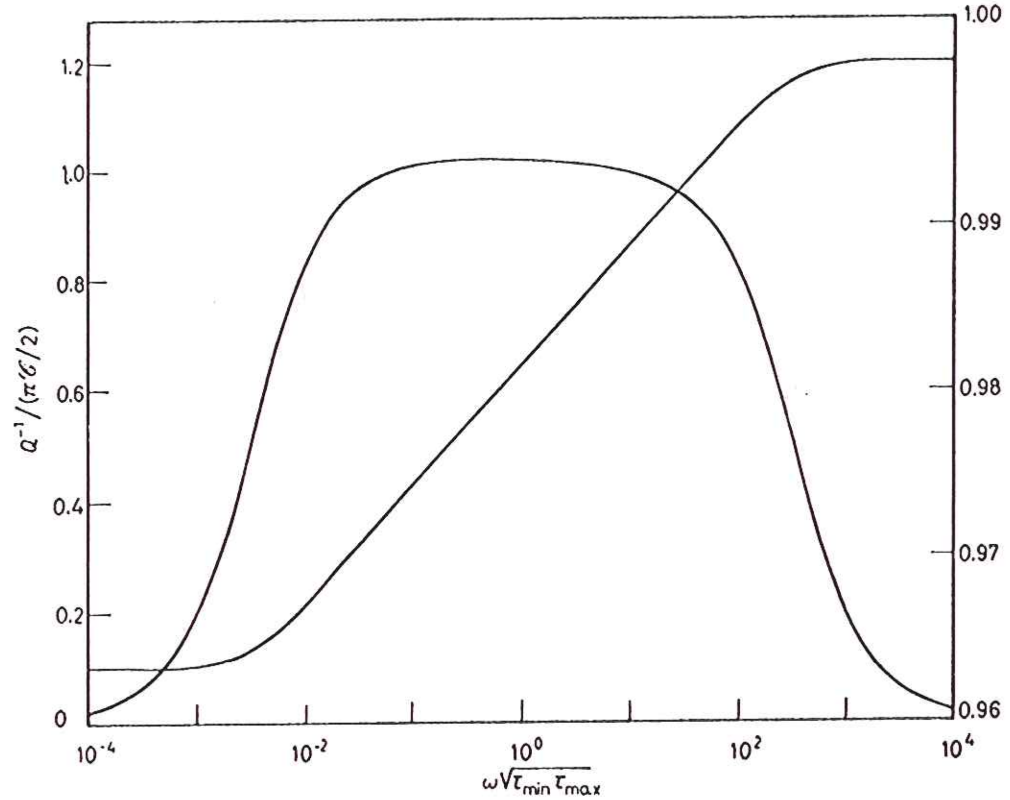


Fig. 3. – $Q^{-1}/(\pi\mathscr{C}/2)$ and $|M|/M_I$ as functions of $\omega\sqrt{\tau_{\min}\tau_{\max}}$, $\tau_{\min}/\tau_{\max} = 10^{-5}$, $\mathscr{C} = 100$.

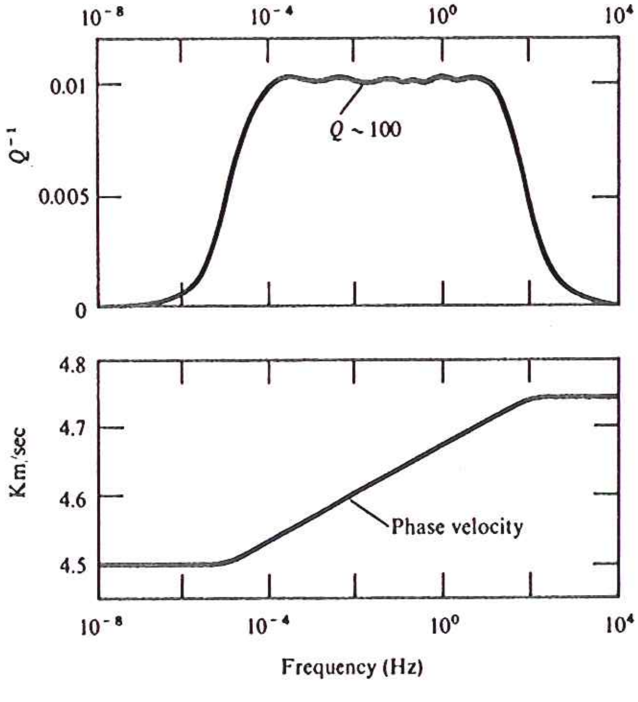


FIGURE **5.15**

Top. Internal friction values. Bottom. Phase velocities obtained (via (5.82)) from a superposition of 12 different relaxation peaks. [After Liu et al., 1976.]

Strictly constant-Q model (Kjartansson 1979)

No anelastic material has $Q(\omega)$ exactly constant for all $0 \le \omega \le \infty$. However, can find such a material if we relax the constraints $0 < M_2 \le M_u < \infty$.

Consider the creep function

$$J(t) = \frac{(\omega_0 t)^{\gamma}}{M_0 \Gamma(1+\gamma)} H(t)$$

↑ gamma function

No instantaneous elastic response, creeps forever.

$J(\nu) = i\nu \int_0^\infty J(t) e^{-i\nu t} dt$ is given by

$$J(\nu) = \frac{1}{M_0} \left(\frac{i\nu}{\omega_0}\right)^{-\gamma}$$

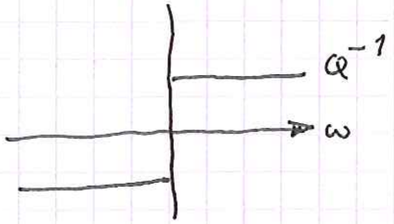
The complex modulus is the reciprocal of this:

$$M(\nu) = M_0 \left(\frac{i\nu}{\omega_0}\right)^{\gamma}$$

On the real axis:

$$M_1(\omega) = M_0 \left|\frac{\omega}{\omega_0}\right|^{\gamma} \sin\left(\tfrac{1}{2}\pi \operatorname{sgn} \omega\right)$$

$$M_2(\omega) = M_0 \left|\frac{\omega}{\omega_0}\right| \cos\left(\tfrac{1}{2}\pi \operatorname{sgn} \omega\right)$$



Thus

$$Q^{-1}(\omega) = \tan\left(\tfrac{1}{2}\pi\gamma\right) \operatorname{sgn} \omega$$

$$\gamma = \frac{1}{\pi} \arctan \frac{1}{Q} \approx \frac{1}{\pi Q}$$

The dispersion law is

$$\frac{M_1(\omega)}{M_1(\omega_0)} = \left(\frac{\omega}{\omega_0}\right)^{\gamma} = \exp\left[\frac{2}{\pi} \arctan\left(\frac{1}{Q}\right) \ln\left(\frac{\omega}{\omega_0}\right)\right]$$

$$\approx 1 + \frac{2}{\pi Q} \ln\left(\frac{\omega}{\omega_0}\right)$$ same as before

Logarithmic dispersion a very robust result — as long as $Q(\omega) \approx$ constant within some band (as observed in the Earth) — insensitive to behavior outside the band.

<u>In summary</u>, we can account for anelasticity within the Earth in seismology be replacing the real elastic bulk & shear moduli by

$$\kappa(\omega)\ [1 + i Q_\kappa^{-1}\ \mathrm{sgn}\,\omega]$$
$$\mu(\omega)\ [1 + i Q_\mu^{-1}\ \mathrm{sgn}\,\omega]$$

↑ bulk and shear <u>quality</u> factors

$$\frac{\kappa(\omega)}{\kappa(\omega_0)} \approx 1 + \frac{2}{\pi Q_\kappa} \ln\left(\frac{\omega}{\omega_0}\right)$$

$$\frac{\mu(\omega)}{\mu(\omega_0)} \approx 1 + \frac{2}{\pi Q_\mu} \ln\left(\frac{\omega}{\omega_0}\right)$$

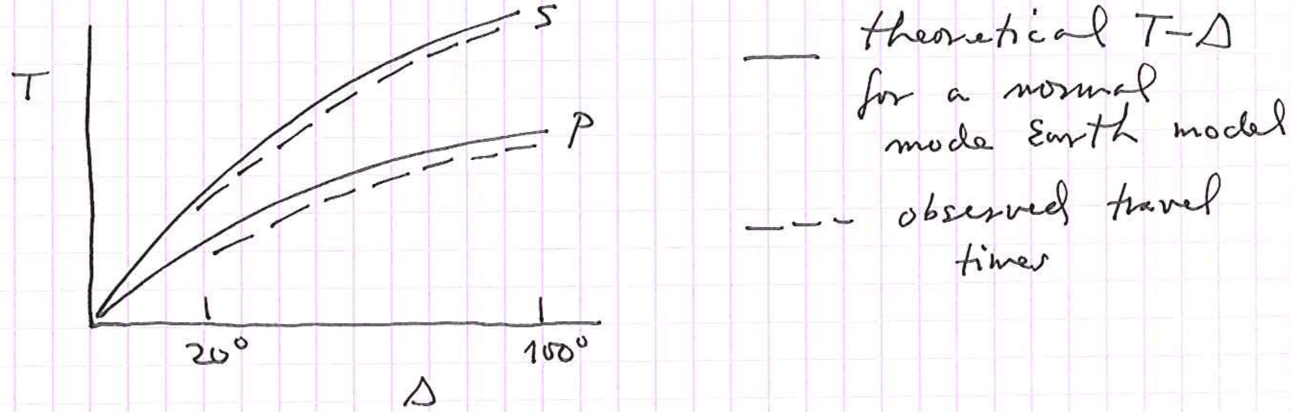
Shear attenuation is much more significant than bulk attenuation so that $Q_\kappa \gg Q_\mu$. A slight amount of bulk dissipation is required somewhere within the Earth to account for the damping of the <u>radial modes</u>. In a polyxtalline material, expect $Q_\kappa / Q_\mu \approx 50-100$ due to local mismatch of elastic moduli at neighboring grains — macroscopic compressive deformation ⇒ microscopic shear (Heinz, Jeanloz, O'Connell 1982).

Average Q_μ in mantle is $Q_\mu^{-1} = \frac{1}{250} \pm 2\%$

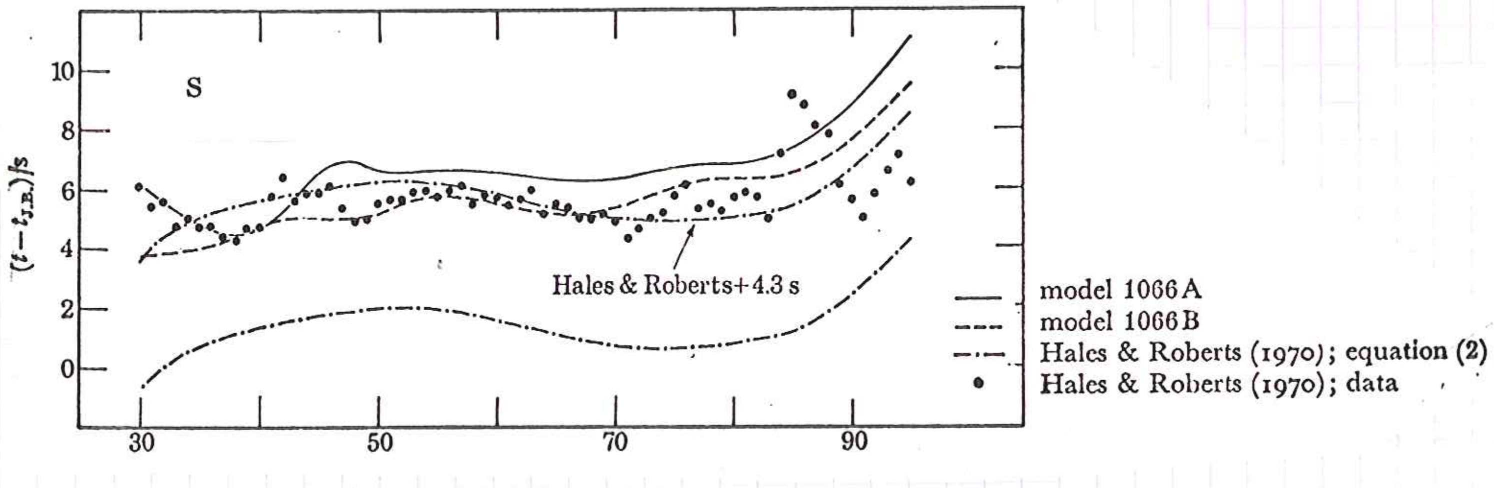
Average in inner core $Q_\mu^{-1} = \frac{1}{110} \pm 25\%$

(Widmer, Masters & Gilbert 1991)

The associated dispersion provides a natural explanation of the travel-time baseline discrepancy. Reliable SNREI models $\rho(r)$, $\kappa(r)$, $\mu(r)$ obtained by fitting normal mode eigenfrequencies first appeared in 1970's.



Observed P times 1-2 s early, observed S times 4-5 s early from 20° to core shadow

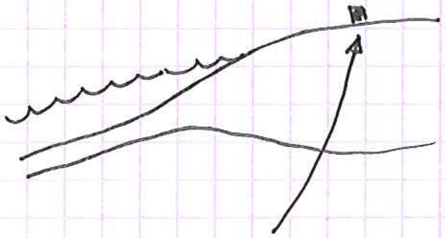


Note the confidence of the modellers (Dziewonski and Gilbert) — they have adjusted the data to get a fit!

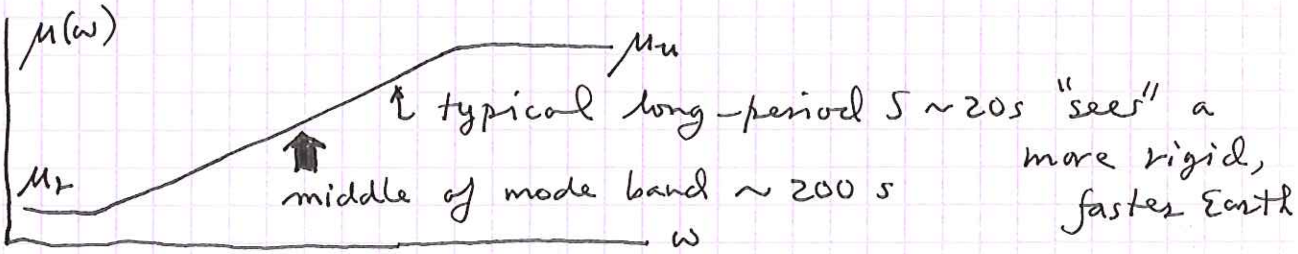
Several rationalizations were suggested:

1. hard to measure absolute travel times because quake location + measuring T is a bootstrap operation.

2. continental bias — all seismic stations on land — modes see a global average



In 1976 Akopyan, Zharkov et al and Liu, Anderson and Kanamori "discovered" dispersion



↑ typical long-period S ~ 20s "sees" a more rigid, faster Earth

⬆ middle of mode band ~ 200 s

$$\frac{\delta\mu}{\mu} \approx \frac{2}{250\pi} \ln\left(\frac{200\,s}{20\,s}\right) \approx 0.005$$

0.5% more rigid at 20s than at 200s.

S velocity $\beta = \sqrt{\mu/\rho}$ $\delta\beta/\beta = \frac{1}{2}\,\delta\mu/\mu = 0.25\%$

Travel time of an S wave at Δ ~ 70° is ~ 20 min. The observed time should be faster by

$$(\cancel{1800}\,\overset{1200}{}\,s)(0.0025) \sim \cancel{4.5}\ 3 \text{ secs}$$ — 2/3 of the baseline discrepancy

The near constancy beyond Δ ~ 20° is due to the fact that the strongest attenuation is in the <u>upper mantle</u>.

Attenuation of normal modes:

Recall the acceleration response of an elastic Earth:

$$a(x,t) = \sum_k M : \varepsilon_k(x_s)\, s_k(x) \cos\omega_k t$$

where we have taken the origin time to be $t_s = 0$ for simplicity.

Anelasticity changes each of the real eigenfrequencies ω_k to:

$$\omega_k \to \omega_k + i\overset{\gamma_k}{\cancel{\alpha_k}} = \omega_k\left(1+\tfrac{1}{2}i\,Q_k^{-1}\right) \qquad \overset{\gamma_k}{\cancel{\alpha_k}} = \frac{\omega_k}{2Q_k}$$

decay rate

mode quality factor

draw picture of Lorentzian

see DT page 234-235

If we ignore the effect ~~[illegible]~~ on the eigenfunctions — this is a good approximation for reasonably well-isolated modes, we can write

$$a(x,t) = \sum_k M : \varepsilon_k(x_s)\, s_k(x) \cos\omega_k t \; \underbrace{e^{-\omega_k t/2Q_k}}_{\text{exponential decay}}$$

discuss in freq. domain also

The mode quality factor may be found using Rayleigh's principle (imaginary perturbation)

$$\delta\omega^2 = 2\omega\delta\omega = \int_\oplus \left[\delta\kappa(\nabla\cdot s)^2 + 2\mu(\delta:\delta) - \delta\rho\,\omega^2 s\cdot s\right] dV$$

$$\delta\kappa = i\kappa(\omega)\,Q_\kappa^{-1} \qquad \delta\omega = i\alpha = \frac{i\omega}{2Q}$$

$$\delta\mu = i\mu(\omega)\,Q_\mu^{-1} \qquad 2\omega\,\delta\omega = i\omega^2/Q$$

$$\frac{1}{Q_k} = \frac{1}{\omega_k^2}\int_\oplus \Big[\underbrace{\kappa(\omega_k)\,Q_\kappa^{-1}(\nabla\cdot s)^2}_{\text{damping due to bulk anelasticity}} + \underbrace{2\mu(\omega_k)\,Q_\mu^{-1}(\delta_k:\delta_k)}_{\text{damping due to shear anelasticity}}\Big]\, dV$$

If we make the (not very good) approximation that Q_κ and Q_μ are <u>constant</u> within $\oplus$:

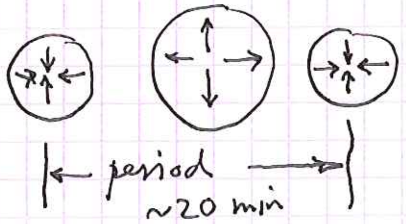
$$\frac{1}{Q_k} \approx \frac{v_c}{Q_\kappa} + \frac{v_s}{Q_\mu}$$ where v_c and v_s are the <u>fractional</u> <u>compressional and shear elastic energy</u> of the mode

$$v_c = \frac{\int_\oplus \kappa \, (\nabla\cdot s)^2 \, dV}{\omega^2 \int_\oplus \rho \, s\cdot s \, dV} \quad , \quad v_s = \frac{\int_\oplus 2\mu \, (\mathbf{d}:\mathbf{d}) \, dV}{\omega^2 \int_\oplus \rho s\cdot s \, dV}$$

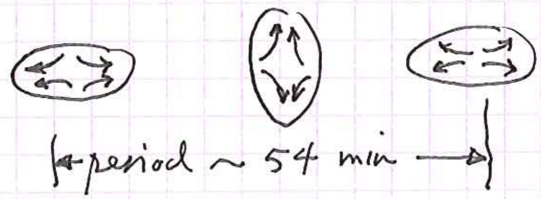
$v_c + v_s = 1$ (or $v_c + v_s + v_g = 1$ when we include the effect of gravity)

A simple example:

${}_0S_0$ (fundamental radial mode)



${}_0S_2$ (fundamental spheroidal mode – "football" mode)



mode	v_c	v_s	v_g		Q
${}_0S_0$	1.304	0.028	−0.332	(destabilizing)	~6000
${}_0S_2$	0.115	0.546	0.339		~300

${}_0S_0$ has <u>very</u> <u>little</u> <u>shear</u> deformation – almost entirely compressional.

The Q of ${}_0S_2$ is not well determined but is about 300. One expects the Q of ${}_0S_0$ to be much higher if $Q_\kappa \gg Q_\mu$.

Roughly:

$$Q({}_0S_0) \approx \underset{\substack{\uparrow \\ Q({}_0S_2)}}{300}\left(\frac{0.546}{0.028}\right) \sim 6000$$

Riedesel et al. (1979) find $Q({}_0S_0) \sim 6000$ from a 2000 hour [redacted] (!) stack of IDA records after the Indonesian earthquake.

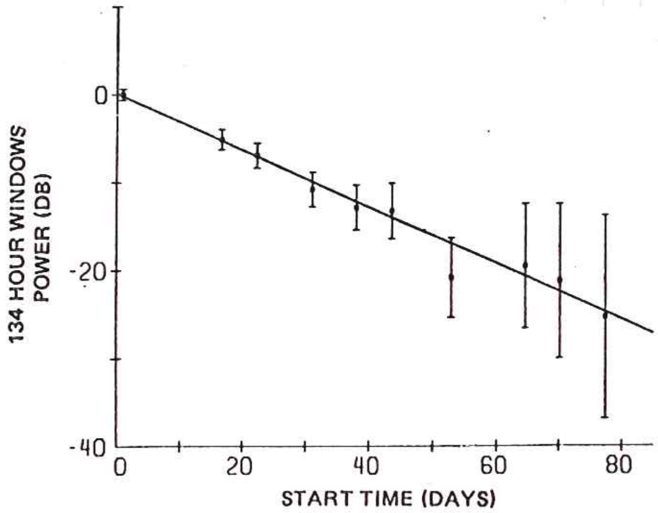


Figure 7. Log integrated power versus time for the weighted stack using windows of 134 hr. The value of *Q* here is 5800. Error bars are obtained using the work of Dahlen (1976). The unweighted least squares procedure was used.

The operators who were installing the KIP station 2 weeks after the event thought they'd picked a noisy site by mistake until they got home and made a spectrum.

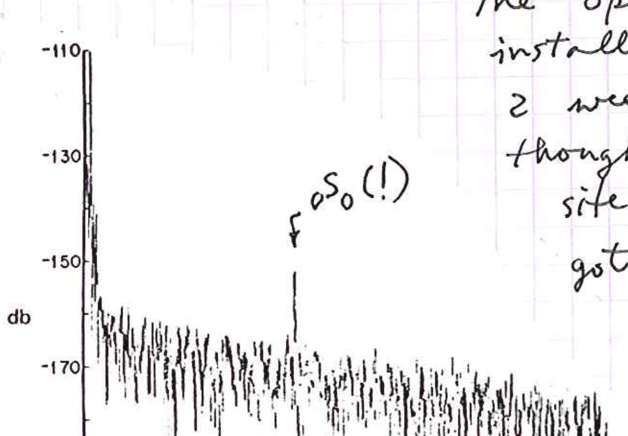


Figure 3. A power spectrum of 120 hr of record from station KIP beginning 18 days after the origin time of the event. Notice the high amplitude of ${}_0S_0$ while all other modes have died away. Station KIP did not start operation until 2 weeks after the earthquake.

Normal Modes of a SNREI Earth

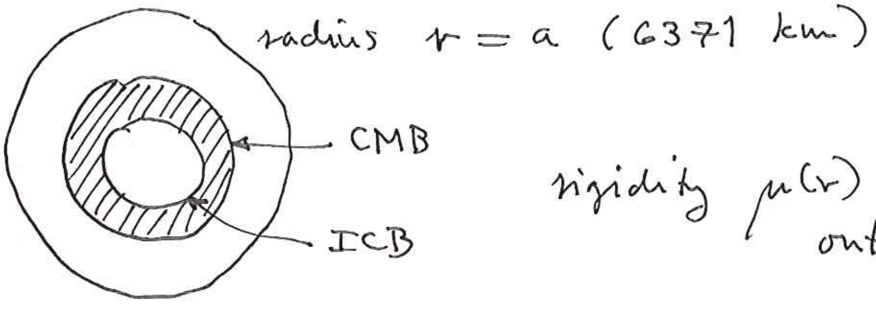
(SNREI = spherical, non-rotating, elastic, isotropic

Characterized by three functions of radius r only:

$\rho(r)$: density

$\kappa(r)$: incompressibility

$\mu(r)$: rigidity



rigidity $\mu(r) = 0$ in fluid outer core between CMB & ICB

To find the modes ω, $s(x)$ we must solve:

$$-\rho\omega^2 \mathbf{s} = \nabla\cdot\mathbf{T} \quad \text{in } \oplus$$

$$\hat{n}\cdot\mathbf{T} = \hat{r}\cdot\mathbf{T} = 0 \quad \text{on } \partial\oplus \qquad (\text{ and } [\hat{r}\cdot\mathbf{T}]^{+}_{-} = 0 \text{ on CMB \& ICB})$$

stress: $\mathbf{T} = \kappa(\nabla\cdot\mathbf{s})\mathbf{I} + 2\mu\boldsymbol{\delta}$

deviatoric strain: $\boldsymbol{\delta} = \frac{1}{2}[\nabla\mathbf{s} + (\nabla\mathbf{s})^T] - \frac{1}{3}(\nabla\cdot\mathbf{s})\mathbf{I}$

$$\operatorname{tr}\boldsymbol{\delta} = 0$$

Boundary-value problem for a second-order PDE

$$\nabla\cdot\mathbf{T} = \nabla\cdot\left[\kappa(\nabla\cdot\mathbf{s})\mathbf{I} + 2\mu\boldsymbol{\delta}\right] \quad \text{i.e.}$$

$$\partial_i T_{ij} = \partial_i\left[\kappa(\partial_k s_k)\delta_{ij} + 2\mu\delta_{ij}\right]$$

$$= \partial_j[\kappa(\partial_k s_k)] + \partial_i(2\mu\delta_{ij})$$

Note that $\kappa(r)$ and $\mu(r)$ appear differentiated.

We seek to reduce this PDE system to an ODE system (PDE = *partial* differential eqn; ODE = *ordinary* differential eqn) in radius r only. This possible because of spherical symmetry of the Earth model — the problem is *separable*.

We begin by expressing s in terms of 3 scalars U, V, W:

$$s = \hat{r}\,U + \nabla_1 V - \hat{r} \times \nabla_1 W$$

The scalars $U(x)$, $V(x)$, $W(x)$ are *unique* provided we require that

$$\underset{\substack{\text{any}\\ \text{spherical}\\ \text{shell}}}{\int V\,dA} = \underset{\text{ditto}}{\int W\,dA} = 0 \qquad \text{i.e. average to zero on every spherical shell}$$

∇_1 is the surface gradient on the *unit* sphere

$$\nabla = \hat{r}\,\partial_r + \frac{1}{r}\nabla_1$$

$$\nabla_1 = \hat{\theta}\,\partial_\theta + \hat{\phi}\,(\sin\theta)^{-1}\,\partial_\phi$$

If V, W are constant functions of θ, ϕ then $\nabla_1 V = \nabla_1 W = 0$ — this is the reason for the restriction.

The three scalars U, V, W can be expanded in *surface spherical harmonics* $Y_l^m(\theta, \phi)$

These are a known orthogonal basis — they are the eigenfunctions of $\nabla_1 \cdot \nabla_1 = \nabla_1^2$ — the surface *Laplacian*:

$$\nabla^2 = \frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\nabla_1^2$$

$$\nabla_1^2 = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2$$

("spherical part" of ∇^2)

$$\nabla_1^2 Y_\ell^m(\theta,\phi) = -\ell(\ell+1)Y_\ell^m(\theta,\phi)$$

They are *orthogonal* — and we *normalize* them — such that

$$\int_\Omega Y_\ell^m Y_{\ell'}^{m'}\,dA = \delta_{\ell\ell'}\delta_{mm'}$$

Ω ← unit sphere $|r|=1$

In *quantum* mechanics it is natural to use *complex* Y_ℓ^m but we shall use *real* ones.

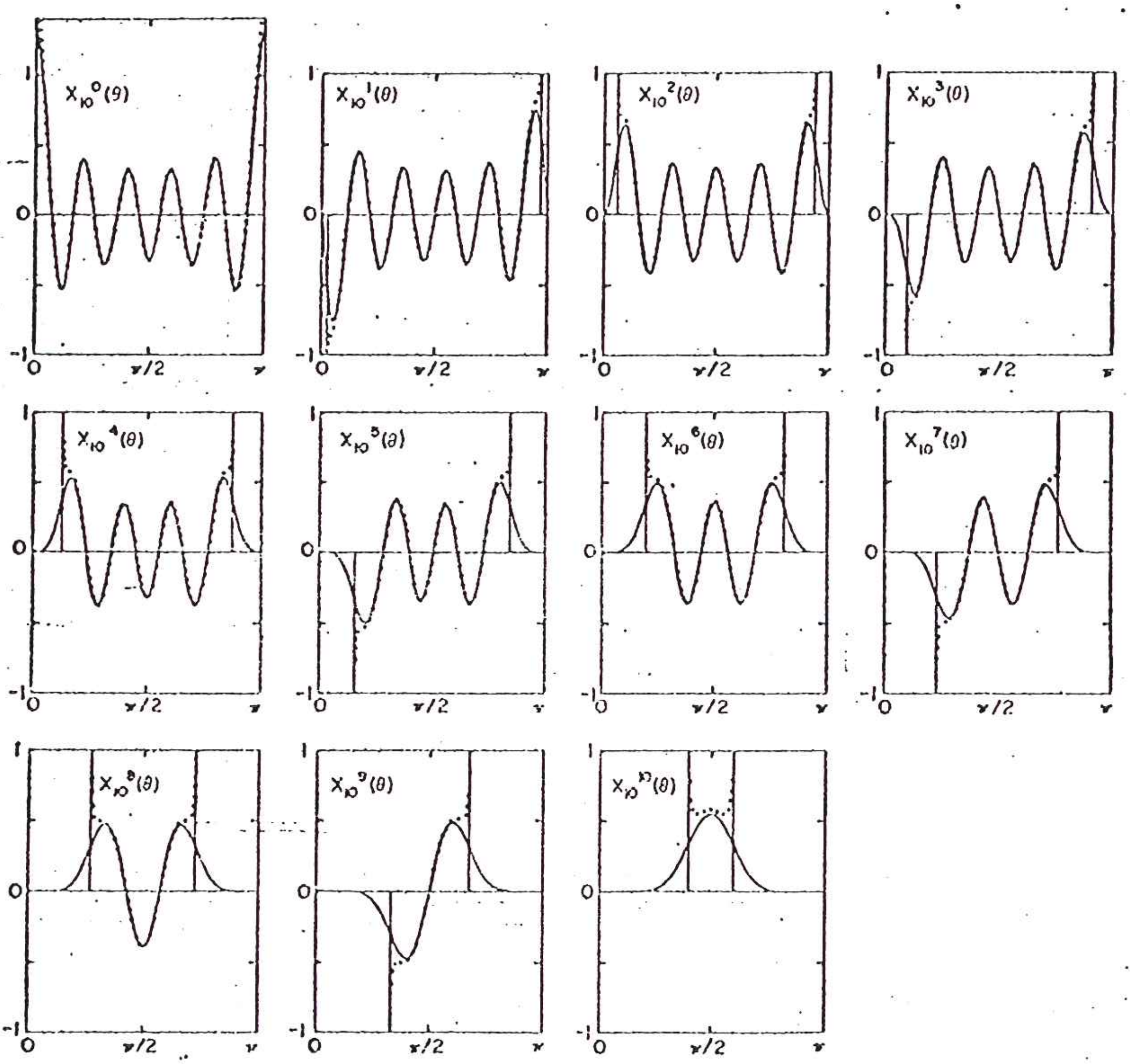
$$Y_\ell^m(\theta,\phi) = X_\ell^m(\theta)\begin{Bmatrix}\sqrt{2}\cos m\phi & \text{if } m>0\\ 1 & \text{if } m=0\\ \sqrt{2}\sin m\phi & \text{if } m<0\end{Bmatrix}$$

$X_\ell^m(\theta)$ has $\ell-|m|$ nodes between 0 and π

$2|m|$ nodes as a function of ϕ

$$X_\ell^{-m}(\theta) = (-1)^m X_\ell^m(\theta)$$

Figure shows $X_\ell^{10}(\theta)$ compared with an asymptotic approximation valid for $\ell \gg 1$.


$X_{10}^{0}(\theta)$
$X_{10}^{1}(\theta)$
$X_{10}^{2}(\theta)$
$X_{10}^{3}(\theta)$
$X_{10}^{4}(\theta)$
$X_{10}^{5}(\theta)$
$X_{10}^{6}(\theta)$
$X_{10}^{7}(\theta)$
$X_{10}^{8}(\theta)$
$X_{10}^{9}(\theta)$
$X_{10}^{10}(\theta)$
1
0
-1
0
$\pi/2$
π

We thus express the displacement in a SNREI model by

$$s(x,t) = \sum_{\ell}\sum_{m} \hat{r}\, U_\ell^m(r)\, Y_\ell^m(\theta,\phi) \qquad (*)$$

$$+ V_\ell^m(r)\, \nabla_1 Y_\ell^m(\theta,\phi) + W_\ell^m(r)\left(-\hat{r}\times\nabla_1 Y_\ell^m(\theta,\phi)\right)$$

Vector spherical harmonics defined by:

$$P_\ell^m = \hat{r}\, Y_\ell^m$$

$$B_\ell^m = \nabla_1 Y_\ell^m$$

$$C_\ell^m = -\hat{r}\times\nabla_1 Y_\ell^m$$

These are orthonormal in the sense

$$\int_\Omega P_\ell^m \cdot P_{\ell'}^{m'}\, dA = \delta_{\ell\ell'}\,\delta_{mm'}$$

$$\int_\Omega B_\ell^m \cdot B_{\ell'}^{m'}\, dA = \ell(\ell+1)\,\delta_{\ell\ell'}\,\delta_{mm'}$$

$$\int_\Omega C_\ell^m \cdot C_{\ell'}^{m'}\, dA = \ell(\ell+1)\,\delta_{\ell\ell'}\,\delta_{mm'}$$

Also any "cross-integral" $\int_\Omega P_\ell^m \cdot B_{\ell'}^{m'}\, dA$, etc. $= 0$.

The divergence of a displacement field of the form $*$ is

$$\nabla\cdot s = \sum_{\ell m}\underbrace{\left[\partial_r U_\ell^m + \frac{2U_\ell^m}{r} - \frac{\ell(\ell+1)V_\ell^m}{r}\right]Y_\ell^m}_{\text{no dependence on } W_\ell^m}$$

And the curl is

$$\nabla \times s = \sum_l \sum_m \left[(l(l+1) \tfrac{1}{r} W_l^m \right] \mathbf{P}_l^m$$

$$+ \left[\partial_r W_l^m + \tfrac{3}{r} W_l^m \right] \mathbf{B}_l^m + \left[\partial_r V_l^m + \tfrac{1}{r} U_l^m - \tfrac{1}{r} V_l^m \right] \mathbf{C}_l^m$$

Nomenclature — a field of form:

$-\hat{r} \times \nabla_1 W$: toroidal

$\hat{r} U + \nabla_1 V$: spheroidal

spheroidal and $\nabla \cdot s = 0$: poloidal — arises in study of ⊕'s magnetic field $\mathbf{B}$.

A list of properties:

Toroidal:
1. purely tangential — no $\hat{r}$ component — no change of shape
2. zero divergence
3. curl is spheroidal

Spheroidal:
1. change of shape
2. non-zero divergence
3. curl is toroidal no radial component

The traction vector $\hat{r} \cdot \mathbf{T}$ is given by

$$\hat{r} \cdot \mathbf{T} = \sum_l \sum_m \hat{r} R_l^m(r) Y_l^m + S_l^m(r) \mathbf{B}_l^m + T_l^m(r) \mathbf{C}_l^m$$

$$\left.\begin{aligned} R &= (\kappa + \tfrac{4}{3}\mu)\partial_r U + (\kappa - \tfrac{2}{3}\mu)\left(\tfrac{2}{r} U - \tfrac{l(l+1)}{r} V\right) \\ S &= \mu\left[\partial_r V + \tfrac{1}{r}(U - V)\right] \end{aligned}\right\} \text{spheroidal}$$

$$T = \mu\left[\partial_r W - \frac{W}{r}\right] \text{— toroidal}$$

Every mode is of the form * for a <u>single</u> value of l and m — the Y_l^m's are <u>uncoupled</u> on a spherical Earth. We seek ODE's for U, V, W. There are two ways to proceed, both of which lead to the same result.

1. <u>BRUTE FORCE</u>:

Substitute $\mathbf{s} = \hat{\mathbf{r}}\, U\, Y_l^m + V\, \boldsymbol{\nabla}_1 Y_l^m + W(-\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 Y_l^m)$

into $-\rho\omega^2 \mathbf{s} = \boldsymbol{\nabla}\cdot\mathbf{T}$

$$\mathbf{T} = \kappa(\boldsymbol{\nabla}\cdot\mathbf{s})\mathbf{I} + 2\mu\mathbf{d}$$

and grind away, using the properties of the Y_l^m's. This involves substantial algebra. The result is two uncoupled sets of equations, one for the toroidal modes and one for the spheroidal modes

Toroidal modes:

$$\frac{d}{dr}\begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} r^{-1} & \mu^{-1} \\ -\rho\omega^2 + \mu(l-1)(l+2)r^{-2} & -3r^{-1} \end{bmatrix}\begin{bmatrix} W \\ T \end{bmatrix}$$

Spheroidal modes:

$$\frac{d}{dr}\begin{bmatrix} U \\ V \\ R \\ S \end{bmatrix} = \begin{bmatrix} -2\lambda\sigma^{-1}r^{-1} & \lambda\sigma^{-1}l(l+1)r^{-1} & \sigma^{-1} & 0 \\ -r^{-1} & r^{-1} & 0 & \mu^{-1} \\ -\rho\omega^2 + 4\gamma r^{-2} & -2l(l+1)\gamma r^{-2} & 2(\lambda\sigma^{-1} - 1)r^{-1} & l(l+1)r^{-1} \\ -2\gamma r^{-1} & -\rho\omega^2 - 2\mu r^{-2} + (\gamma+\mu)l(l+1)r^{-2} & -\lambda\sigma^{-1}r^{-1} & -3r^{-1} \end{bmatrix}\begin{bmatrix} U \\ V \\ R \\ S \end{bmatrix}$$

where $\sigma = \kappa + \frac{4}{3}\mu$, $\gamma = \frac{3\mu\kappa}{\sigma}$, $\lambda = \kappa - \frac{2}{3}\mu$

The associated boundary conditions are:

toroidal: $T = 0$ on free surface, CMB and ICB

spheroidal: $S = 0$ on free surface, CMB and ICB

$R = 0$ on free surface

$[R]_-^+ = 0$ on CMB & ICB.

$$\hat{r}\cdot T = \underbrace{R\hat{r}Y_\ell^m}_{\text{radial comp.}} + S\nabla_1 Y_\ell^m + T(-\hat{r}\times\nabla_1 Y_\ell^m)$$

Method 2 – derive from Rayleigh's principle.

Substitute $*$ into the action

$$I = \frac{1}{2}\int_\oplus [\omega^2\rho s\cdot s - \kappa(\nabla\cdot s)^2 - 2\mu(\tilde{d}:\tilde{d})]\,dV$$

and use orthonormality of Y_ℓ^m's. Result is a new action of the form

$$I = \frac{1}{2}\sum_\ell\sum_m \int_0^a L_\ell^m(r)\,\underset{\uparrow \text{ Jacobian}}{r^2}\,dr$$

— again no coupling — each L_ℓ^m depends only on $U_\ell^m, V_\ell^m, W_\ell^m$

Get a separate action for each spheroidal & toroidal mode.

Toroidal mode:

$$I = \underbrace{\frac{1}{2}\ell(\ell+1)\int_0^a [\rho\omega^2 W^2}_{\substack{\text{kinetic}\\ \text{energy}}} \underbrace{- \mu\left(\partial_r W - \frac{W}{r}\right)^2 - \mu(\ell-1)(\ell+2)\frac{W^2}{r^2}]\,r^2\,dr}_{\substack{\text{(shear) elastic potential}\\ \text{energy of mode}}}$$

Spheroidal mode:

$$I = \frac{1}{2}\int_0^a \Big[\underbrace{\rho\omega^2\left(U^2 + l(l+1)V^2\right)}_{\text{kinetic energy}} - \underbrace{\kappa\left(\partial_r U + \frac{1}{r}\left(2U - l(l+1)V\right)\right)^2}_{\text{compressional elastic energy}}$$

$$\underbrace{- l(l+1)\mu\left(\left(\partial_r V + \frac{U}{r} - \frac{V}{r}\right)^2 + (l-1)(l+2)\frac{V^2}{r^2}\right)}_{\text{shear energy density}} \Big] r^2\, dr$$

shear energy density – left out a term – see page 62 for correct expression

The toroidal Lagrangian is of the form

$$I_{\text{toroidal}} = \int_b^a L\left(W, \frac{dW}{dr}\right) dr$$

$b \leftarrow$ CMB radius – for mantle toroidal modes

Rayleigh's principle asserts that

$$\delta I = \int_b^a \left[\delta W\, \partial_W L + \frac{d}{dr}(\delta W)\, \partial_{dW/dr} L \right] dr$$

$$= \underbrace{\Big[\delta W\, \partial_{dW/dr} L \Big]_b^a}_{\substack{\text{this gives} \\ \text{b.c. } T(b)=T(a)=0}} + \underbrace{\int_b^a \delta W \left[\partial_W L - \frac{d}{dr}\left(\partial_{dW/dr} L\right)\right] dr}_{\substack{\text{this gives} \\ \text{same eqn as} \\ \text{before – one} \\ \text{second-order} \\ \text{ODE}}}$$

Ditto for the spheroidal modes

$$I_{\text{spheroidal}} = \int_0^a L\left(U, V, \frac{dU}{dr}, \frac{dV}{dr}\right) dr$$

In this case get two second-order ODE's and 2 b.c. – one from each of δU and δV – final results same as by BRUTE FORCE method.

Consider a simple example — homogeneous sphere

ρ, κ, μ constant — a marble or shotput or ball bearing

The 2$^{\underline{d}}$ order ODE governing the toroidal modes is

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\ell(\ell+1)}{r^2} + \frac{\omega^2}{\beta^2}\right]W(r) = 0$$

b.c. $T(a) = \mu\left(\frac{dW}{dr} - \frac{W}{r}\right)_a = 0$, where $\beta = \sqrt{\mu/\rho}$ shear wave speed

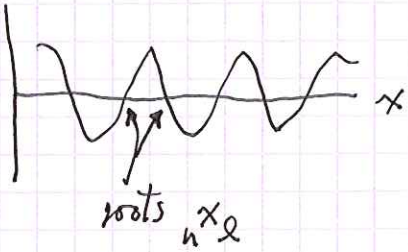
Solutions are the spherical Bessel functions:

$$W(r) = j_\ell\left(\frac{\omega r}{\beta}\right)$$

$\leftarrow$ derivative

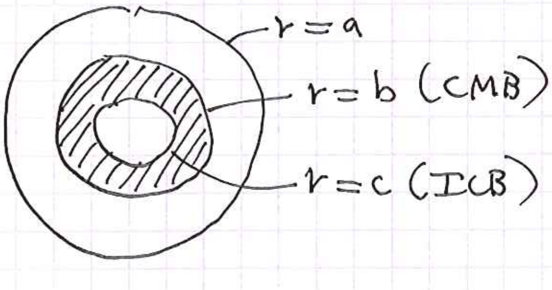
The b.c. $\Rightarrow$ $\left(\frac{\omega a}{\beta}\right) j_\ell'\left(\frac{\omega a}{\beta}\right) - j_\ell\left(\frac{\omega a}{\beta}\right) = 0$

Plot of $x j_\ell'(x) - j_\ell(x)$



Eigenfrequencies ${}_n\omega_\ell = {}_n x_\ell\,(\beta/a)$.

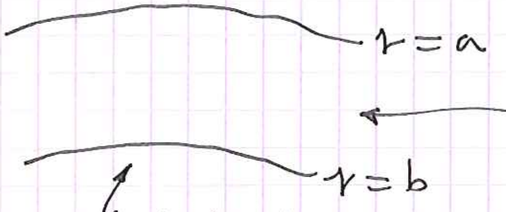
More generally, for a realistic $\oplus$ model, we use numerical integration. There can be no toroidal motion in the fluid core where $\mu = 0$ (More precisely, any $W(r)$ confined to the fluid core is a trivial solution with associated eigenfrequency $\omega = 0$ and stress $T = \mu\left(\frac{dW}{dr} - \frac{W}{r}\right) = 0$).



The toroidal modes of the solid inner core are of little interest — can neither be excited by 'quakes nor observed on the surface. Consider the mantle modes.

$$\frac{d}{dr}\begin{bmatrix}W\\T\end{bmatrix} = \begin{bmatrix}2\times2\text{ matrix}\end{bmatrix}\begin{bmatrix}W\\T\end{bmatrix}\ ; \quad T(b) = T(a) = 0$$

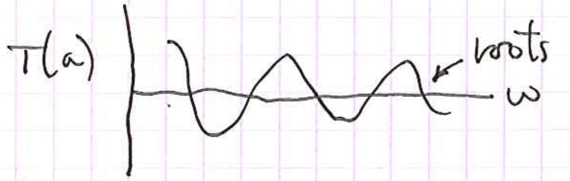
↖ depends on ω — also nice feature does not depend on $d\rho/dr$, etc. — no need to numerically differentiate model — also no dependence on κ, only μ — also depends on ℓ.



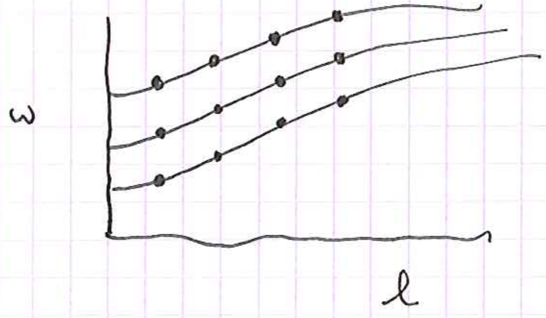
integrate 2x2 to surface $r=a$ using Runga-Kutta integration — use a <u>trial</u> value of ω for a given ℓ

start here with solution $\begin{bmatrix}W\\T\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

Now plot $T(a)$ versus ω:



Conventional to plot results on a <u>dispersion</u> <u>diagram</u>.

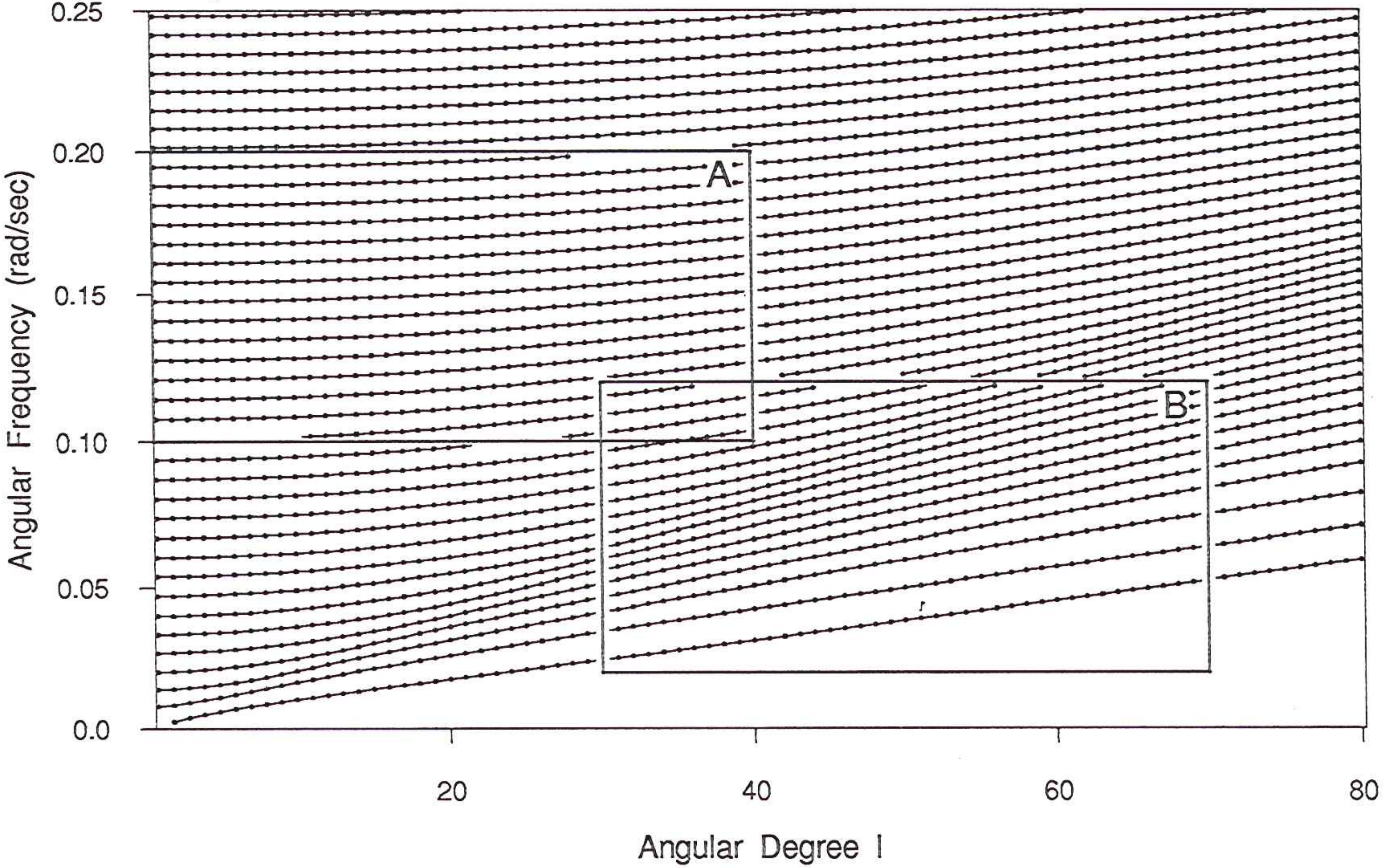


Each dot is a eigenfrequency of the Earth. The 2x2 matrix does not depend on m only ℓ — hence each frequency is $2\ell+1$-<u>degenerate</u>.

The associated eigenfunctions are of form

$$s = {}_nW_\ell(r)\left[-\hat{r}\times\nabla_1 Y_\ell^m(\theta,\phi)\right], \quad \underbrace{-\ell\le m\le\ell}_{2\ell+1\text{ modes}}$$

Ignore the boxes A & B — note the regular pattern



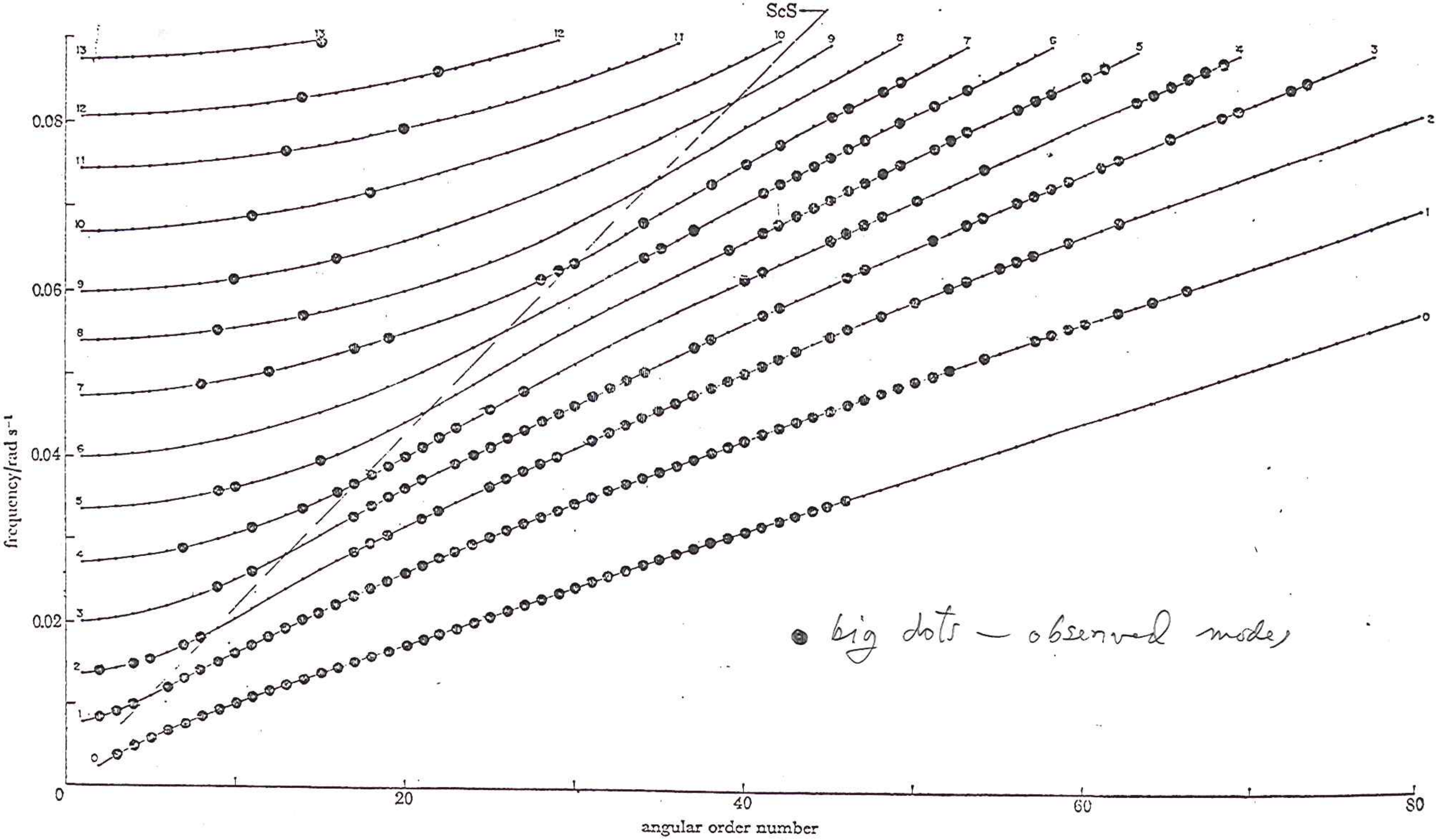
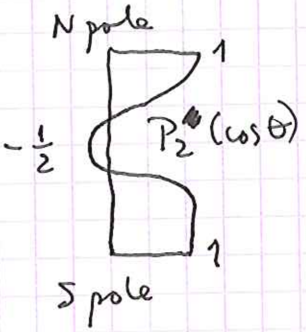


Figure 16. Toroidal normal modes in the (ω, l) plane. The large dots indicate observed modes used in the inversions. Most of the toroidal overtones identified by Brune & Gilbert (1964) fall outside the range of the figure. The dashed line designated 'ScS' divides the modes into two groups according to the normal mode–body wave analogy: modes to the left of this line correspond to $(ScS)_H$ reflexions, those to the right correspond to mantle S_H waves.

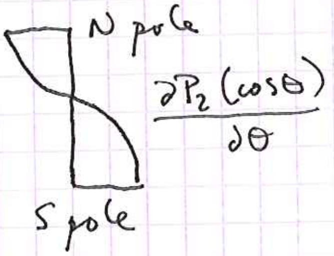
$_0T_2$ (period $\frac{2\pi}{\omega} \approx 44$ mins) is the gravest toroidal mode — not observed until ~2 years ago, following Macquarie Ridge quake — by Walter Zürn & colleagues.

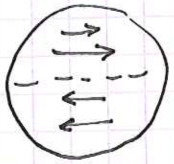
For $m=0$: $Y_2^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_2(\cos\theta)$

← Legendre polynomial



$C_2^0 = -\hat{\phi}\,\frac{\partial Y_2^0}{\partial\theta}$

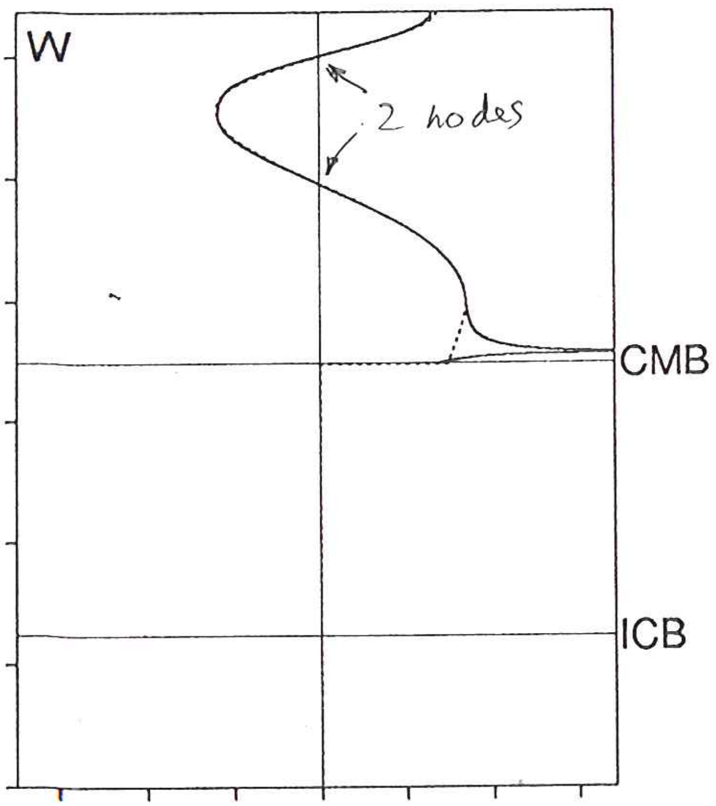




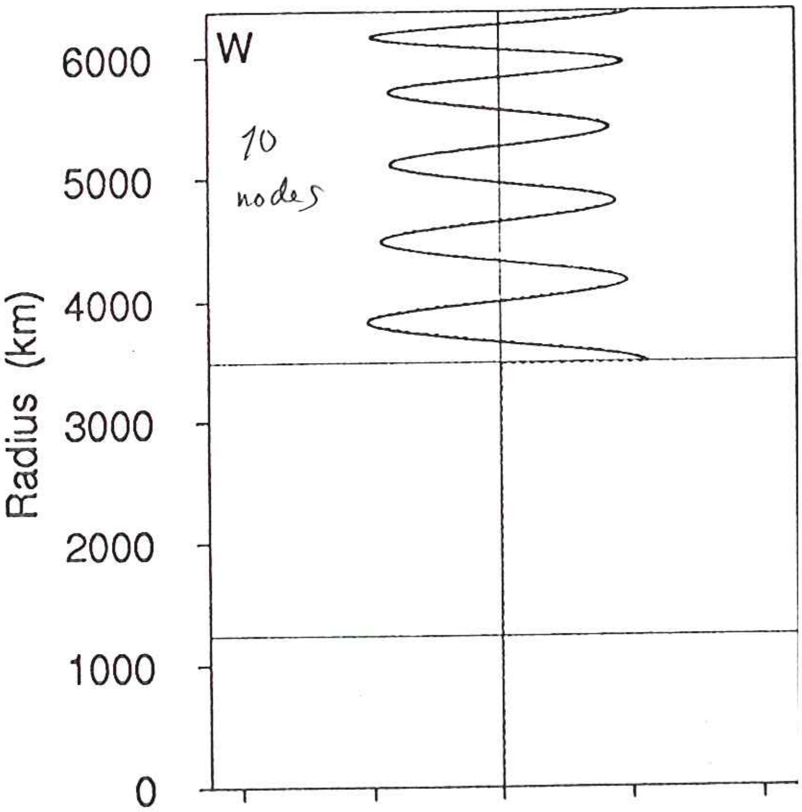
twisting or shearing motion like this.

Every radial eigenfunction $_nW_l(r)$ has n radial nodes — this a property of Sturm–Liouville system. $_2W_{10}$ and $_{10}W_{10}$ shown below — ignore the (asymptotic) solid curves for now — model is 1066A.

Toroidal Mode $_2T_{10}$



Toroidal Mode $_{10}T_{10}$



Spheroidal modes: more generally if we consider the gravitational restoring force as well as the elastic force we must solve a 6 × 6 system:

$$\frac{d}{dr}\begin{bmatrix}u\\ V\\ R\\ S\\ \phi\\ \psi\end{bmatrix} = \begin{bmatrix}\text{6×6 matrix}\\ \text{no depence on}\\ \rho', \kappa', \mu';\\ \text{depends on } l\\ \text{and } \omega\end{bmatrix}\begin{bmatrix}u\\ V\\ R\\ S\\ \phi\\ \psi\end{bmatrix}\begin{array}{l} \\ \\ \\ \\ \swarrow \text{perturbation in grav. potential}\\ \leftarrow \frac{d\phi}{dr} + 4\pi G\rho u + \frac{l+1}{r}\phi\end{array}$$

Adding gravity adds 2 eqns — Poisson's equation $\nabla^2\phi = 4\pi G\,\delta\rho$ is secon-order.

Now three b.c.:

$$\left.\begin{array}{l}R(a) = 0\\ S(a) = 0\end{array}\right\}\text{ traction}$$

$$\psi(a) = 0 \leftarrow \text{gravity}$$

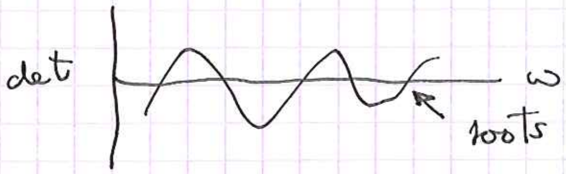
There are two special cases — in the fluid core the 6×6 system reduces to a 4×4 (since $\mu = 0$). For the radial modes ($l = 0$) reduces to a 2×2. To find ${}_n\omega_l$ we start integration at of Earth where there are three linearly independent solutions that are finite in the limit $r \to 0$. Integrate using R–K each of the three to the surface. The general solution must be a linear combination:

$$a\begin{bmatrix} \\ \\ \end{bmatrix}_{I} + b\begin{bmatrix} \\ \\ \end{bmatrix}_{II} + c\begin{bmatrix} \\ \\ \end{bmatrix}_{III}$$

We want (at $r = a$):

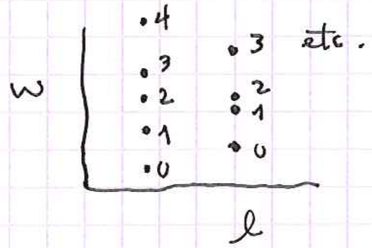
$$\begin{pmatrix}R(a)\\ S(a)\\ \psi(a)\end{pmatrix} = \begin{bmatrix}R_I & R_{II} & R_{III}\\ S_I & S_{II} & S_{III}\\ \psi_I & \psi_{II} & \psi_{III}\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

A solution exists only if det $[3\times3] = 0$.

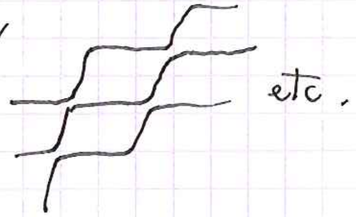


The dispersion diagram ($\omega - l$) is much more complicated for a realistic ⊕ model with a solid inner core & fluid outer core than for the toroidal modes.

Nomenclature: for each degree l of Y_l^m one just counts up from the bottom to determine the radial index or overtone number n

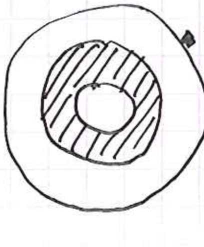


When all the ${}_n\omega_l$ for fixed n are connected the diagram appears to be "terraced"



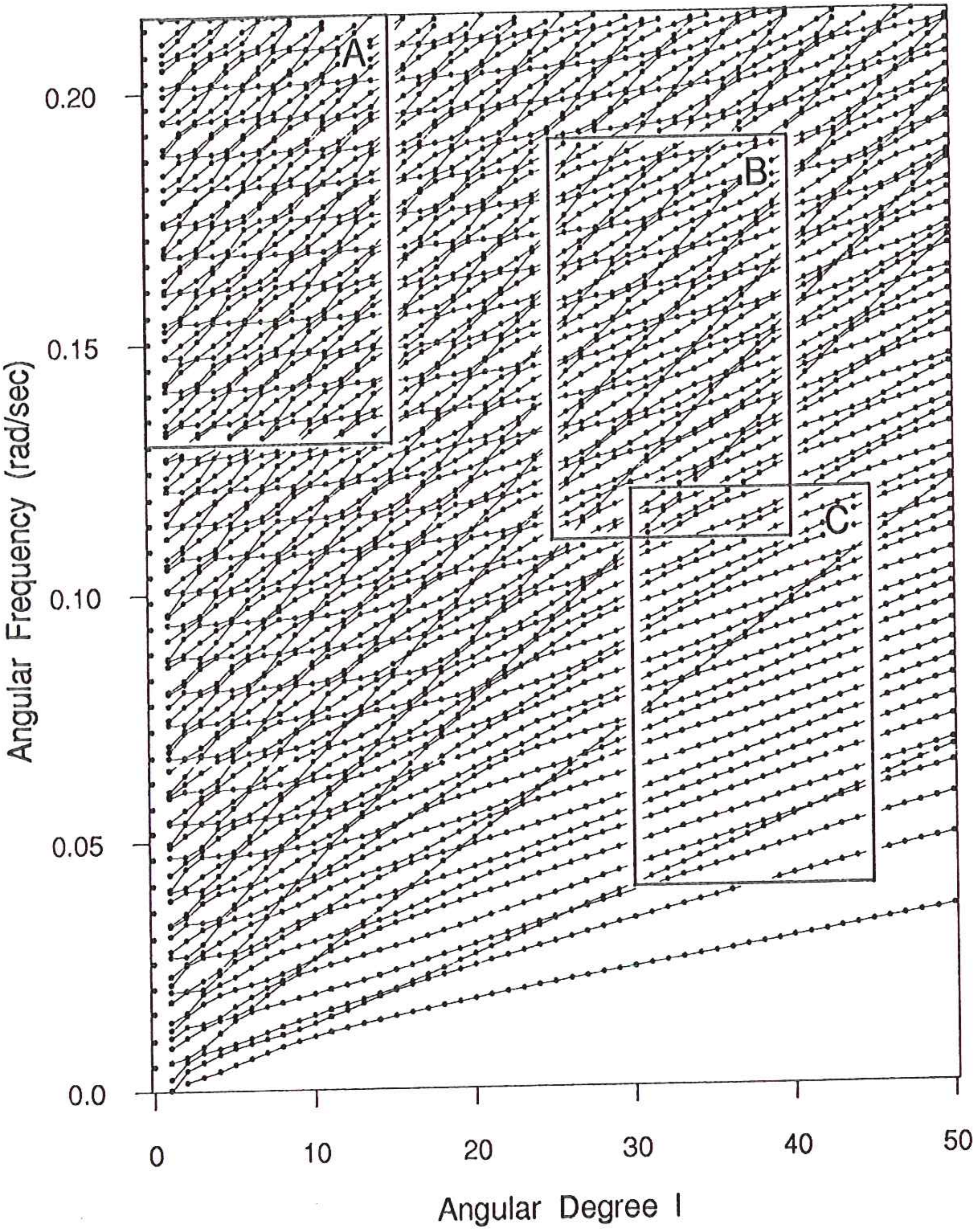
The gravest observed mode is ${}_0S_2$ — a quintuplet — the "football mode" with a period of ~ 54 minutes 
It is visibly split by the Earth's rotation.

The true gravest elastic mode is ${}_1S_1$ — a triplet — the "Slichter mode" — corresponds to 3 rigid-body translations of solid inner core in fluid — period about 4 hours. The recently claimed "detection" by the Canadian superconducting gravimeter is WRONG!



can in principle be detected by a gravimeter at surface — but hard to excite by quakes

Ignore Boxes A, B, C



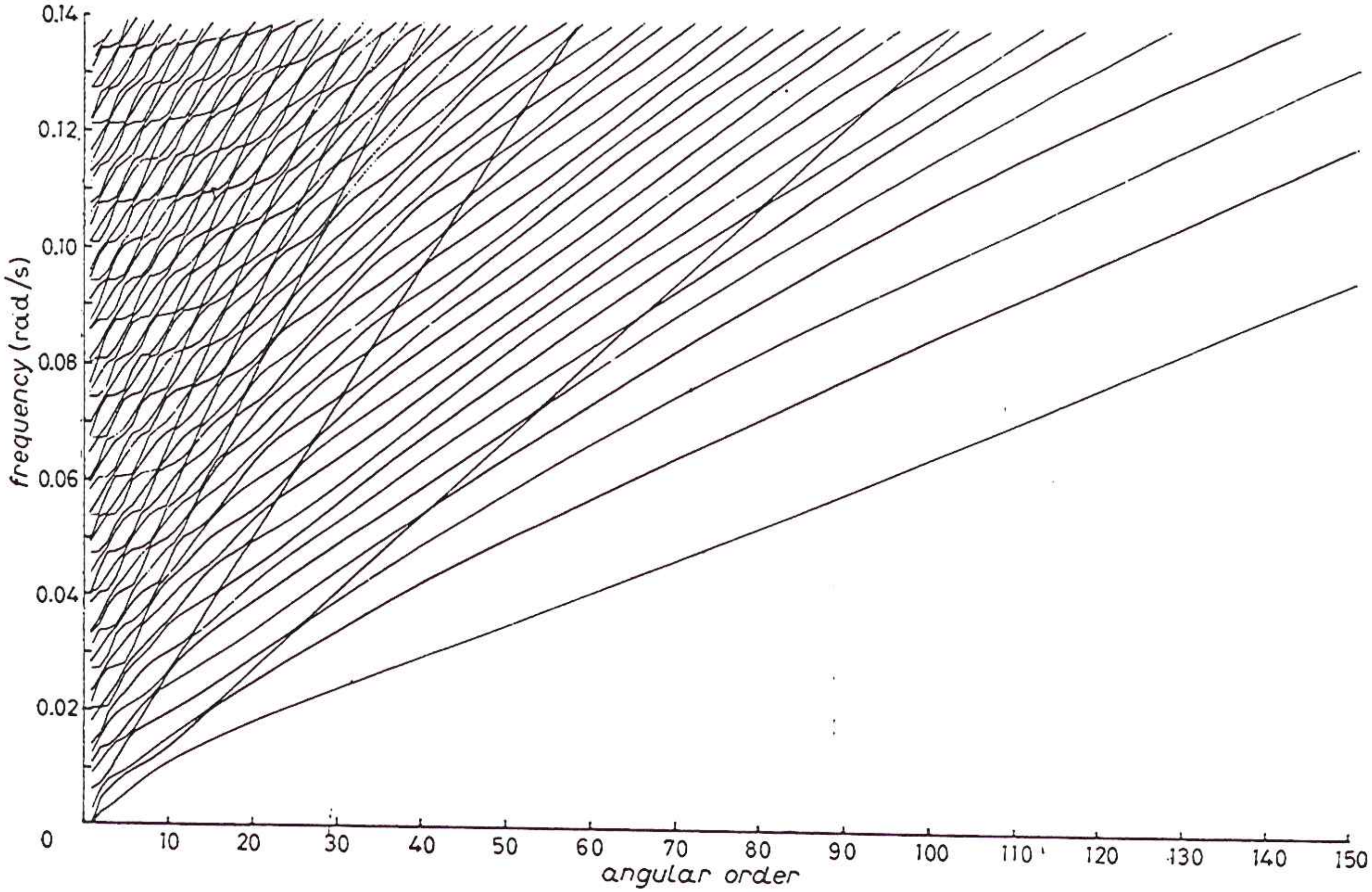


Fig. 3. – (ω, l) diagram for spheroidal, ${}_nS_l$, modes. The points ${}_n\omega_l$ are connected by a continuous line for constant n.

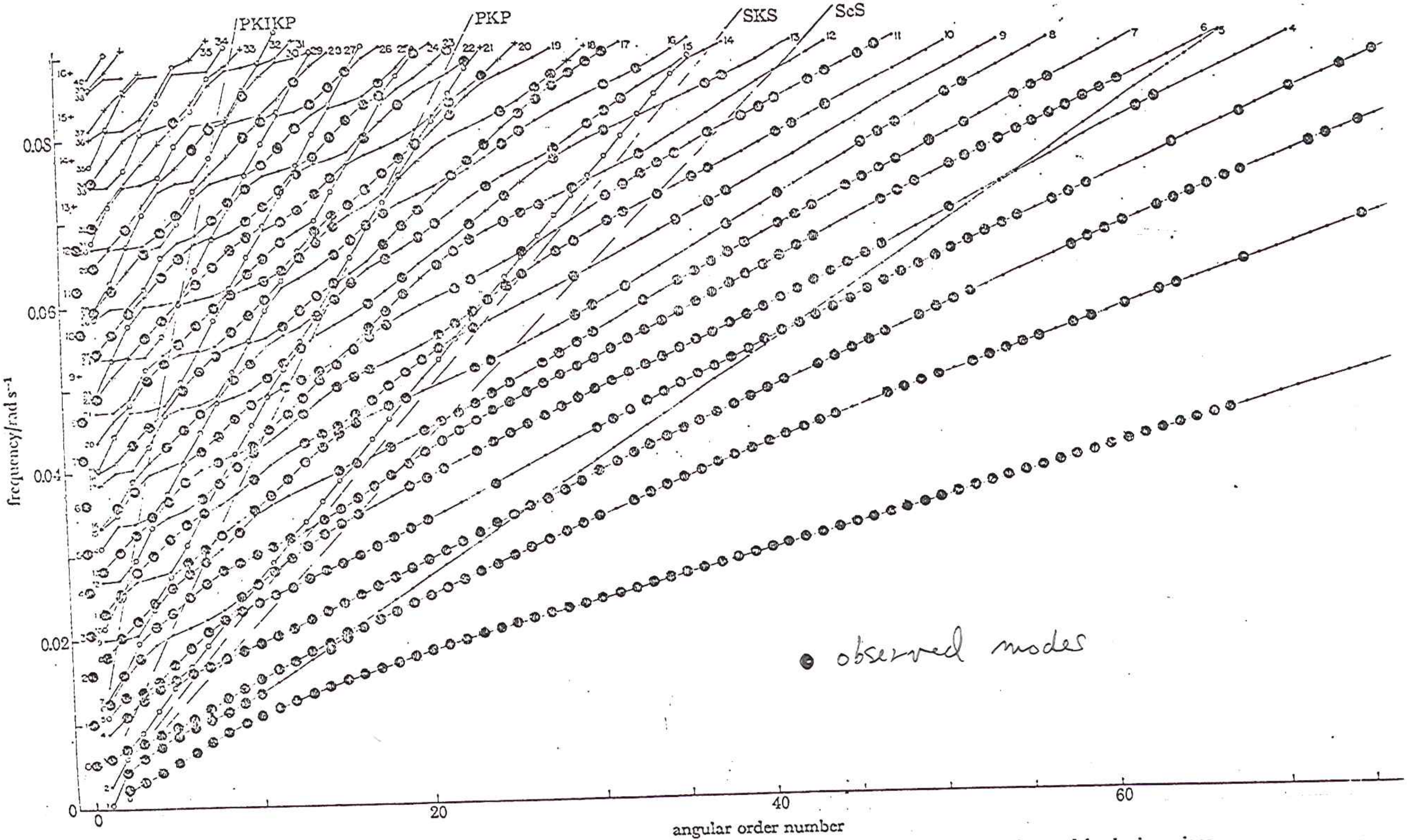
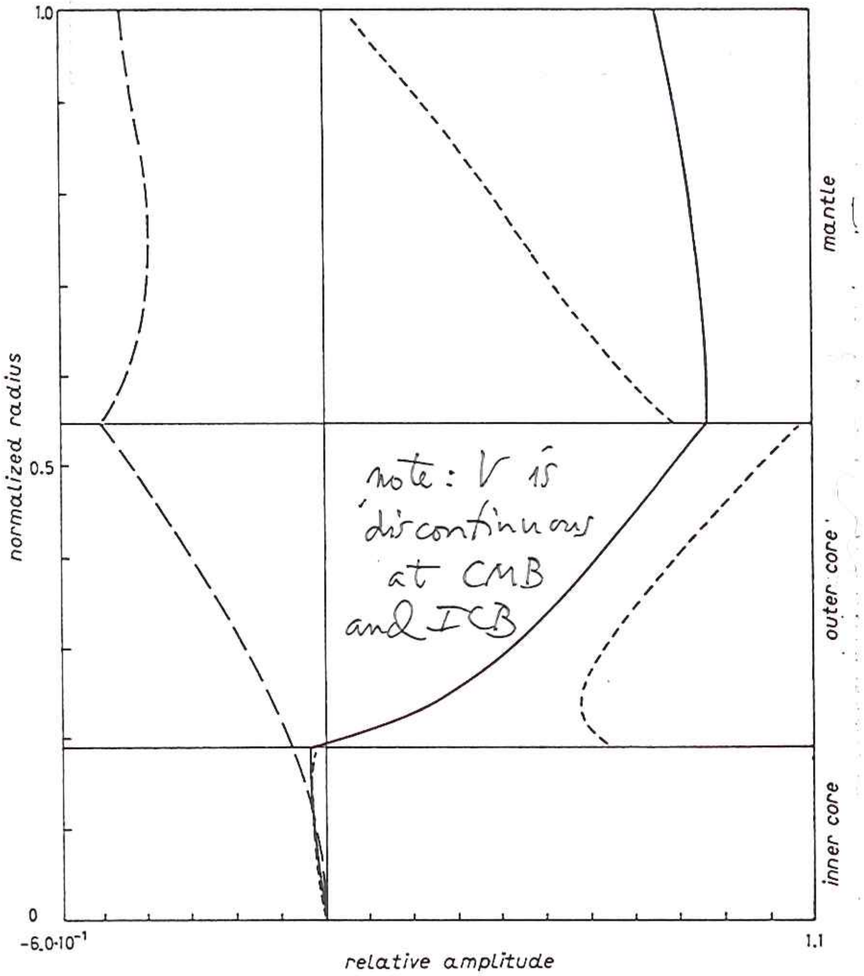
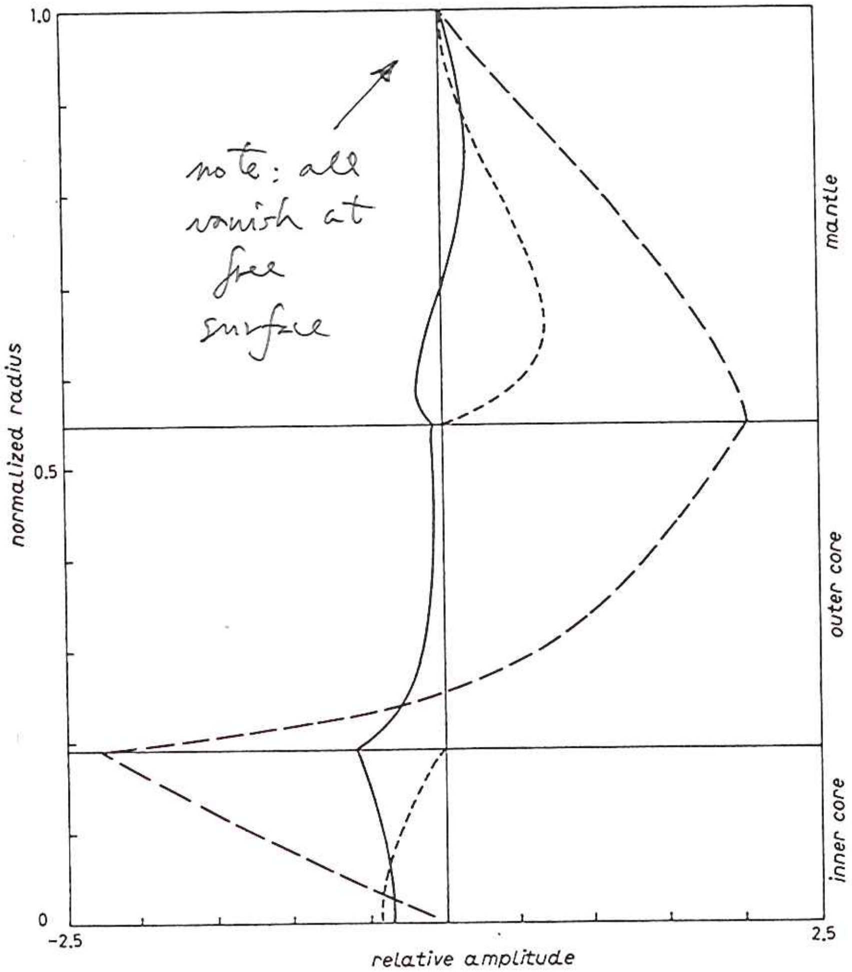


FIGURE 17. Spheroidal normal modes in the (ω, l) plane. The large dots indicate observed modes used in the inversions. For further details we refer the reader to §3 of Alaska II. •, $CE < 0.5$; +, $CE \geqslant 0.5$; ○ core modes.



note: V is discontinuous at CMB and ICB

Fig. 4. – The mode ${}_0S_2$: displacement scalars U (——) and V (– – –), and perturbation potential Φ (— — —).



note: all vanish at free surface

Fig. 5. – The mode ${}_0S_2$: radial-traction scalars R (——) and S (– – –), and scalar ψ (— — —).

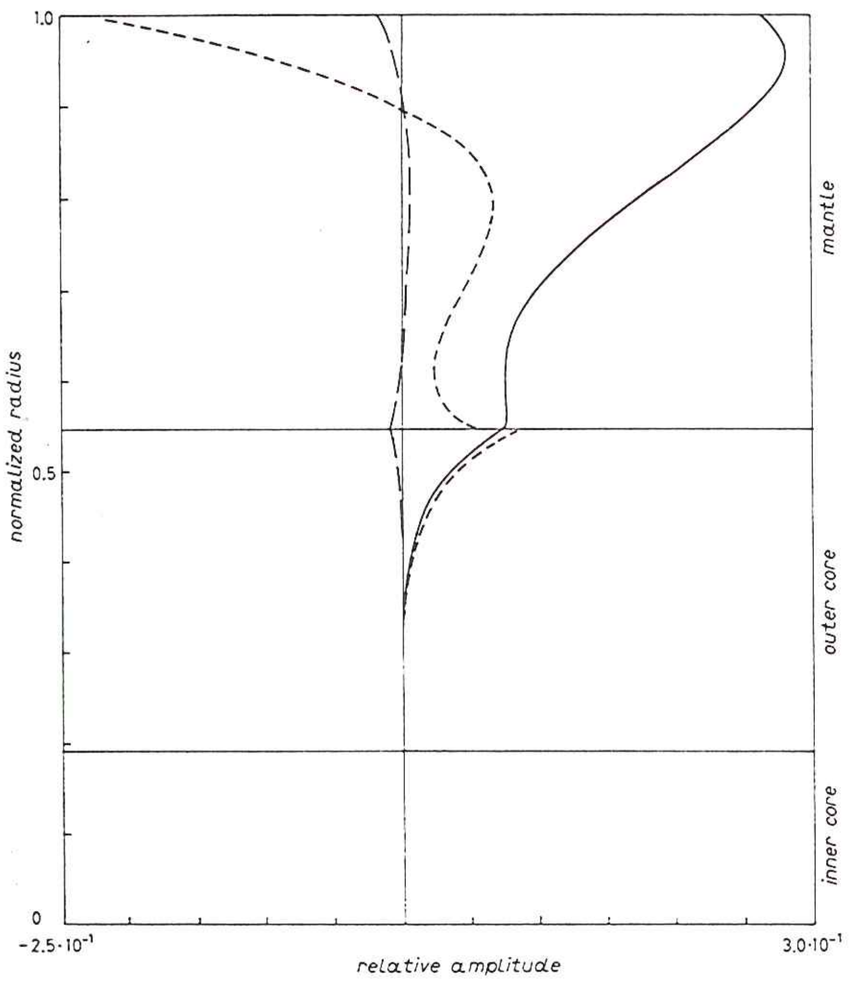
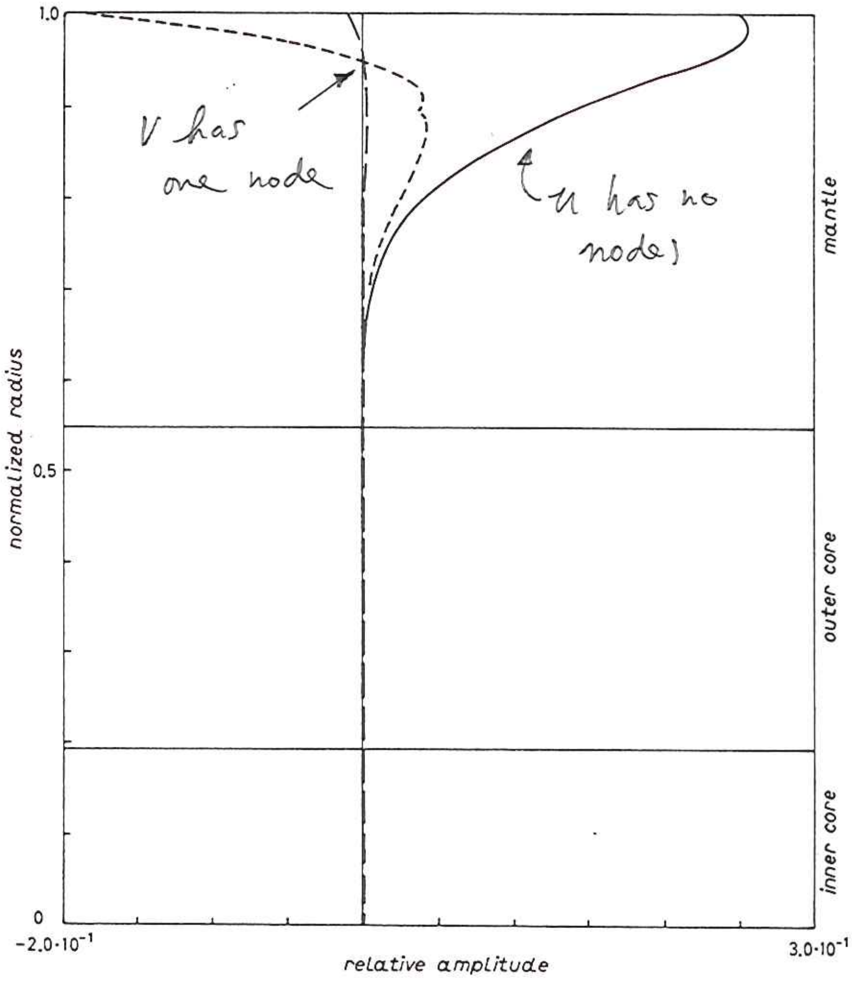


Fig. 7. – The mode $_0S_{10}$: displacement scalars U (———) and V (‐ ‐ ‐), and perturbation potential Φ (— — —).



V has one node

U has no node)

Fig. 10. – The mode $_0S_{19}$: displacement scolars U (———) and V (‐ ‐ ‐), and perturbation potential Φ (— — —).

Surface wave modes

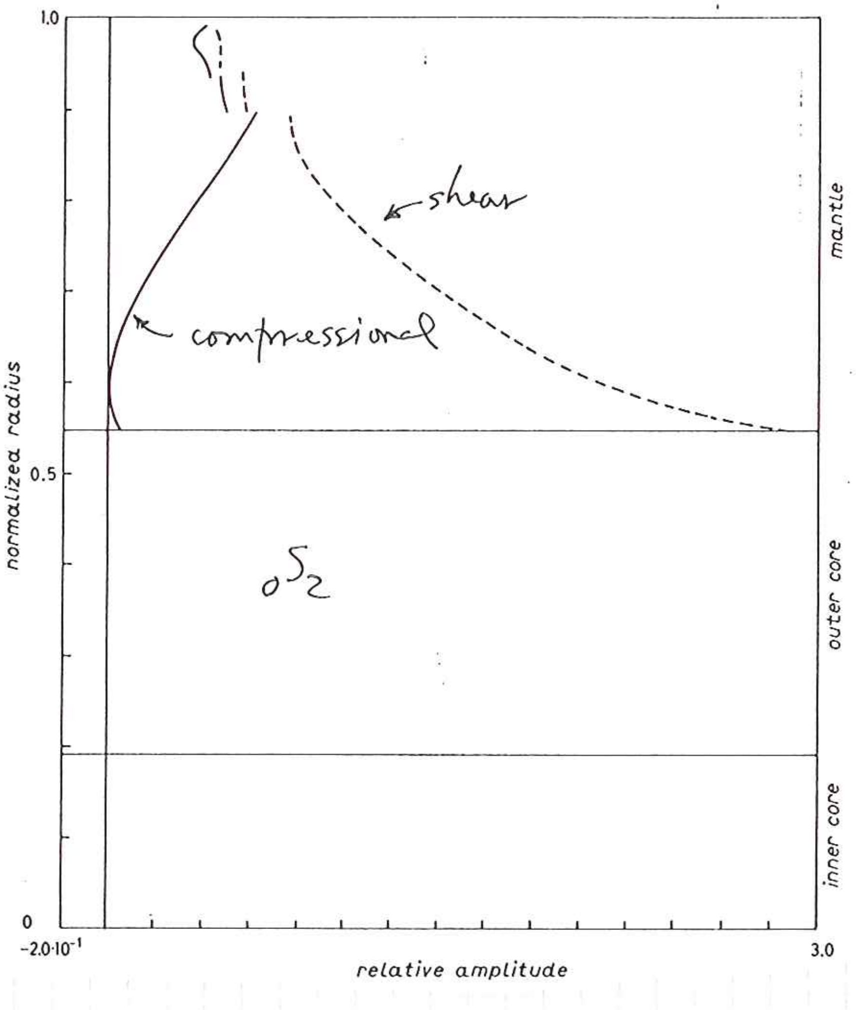
Recall the compressional and shear energy densities are given by

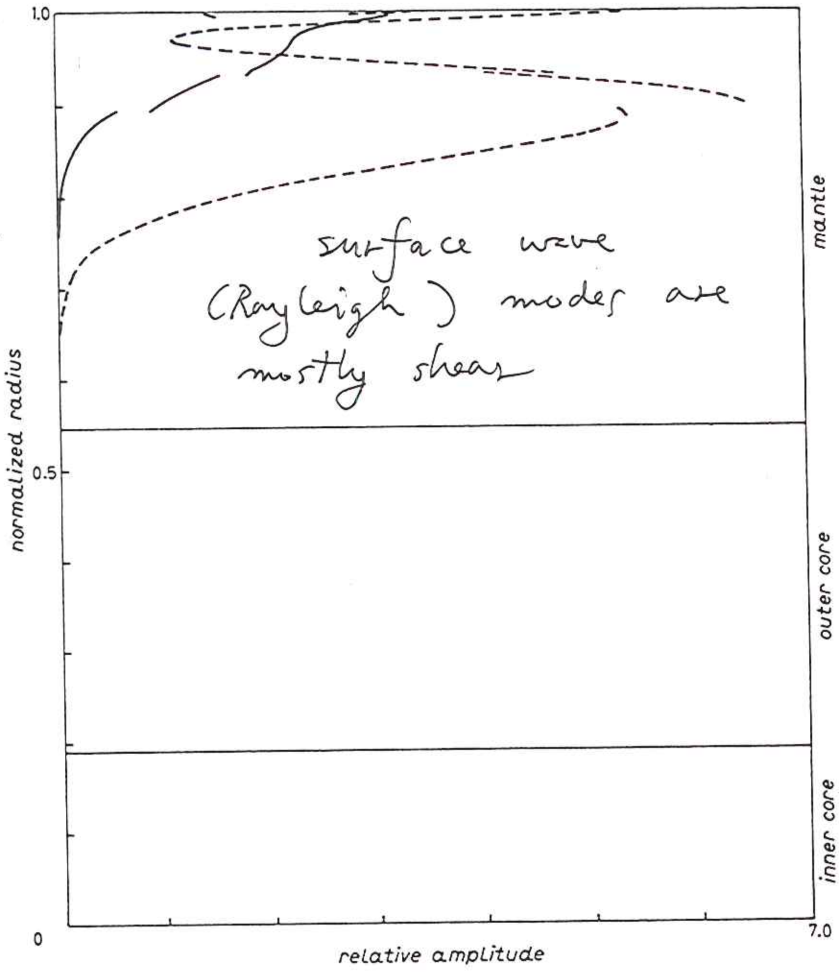
compressional:

$$V_c = \frac{1}{2}\int_0^a \underbrace{\kappa \left[\partial_r U + \frac{1}{r}\left(2U - l(l+1)V\right)\right]^2 r^2}_{\text{density}} dr$$

shear

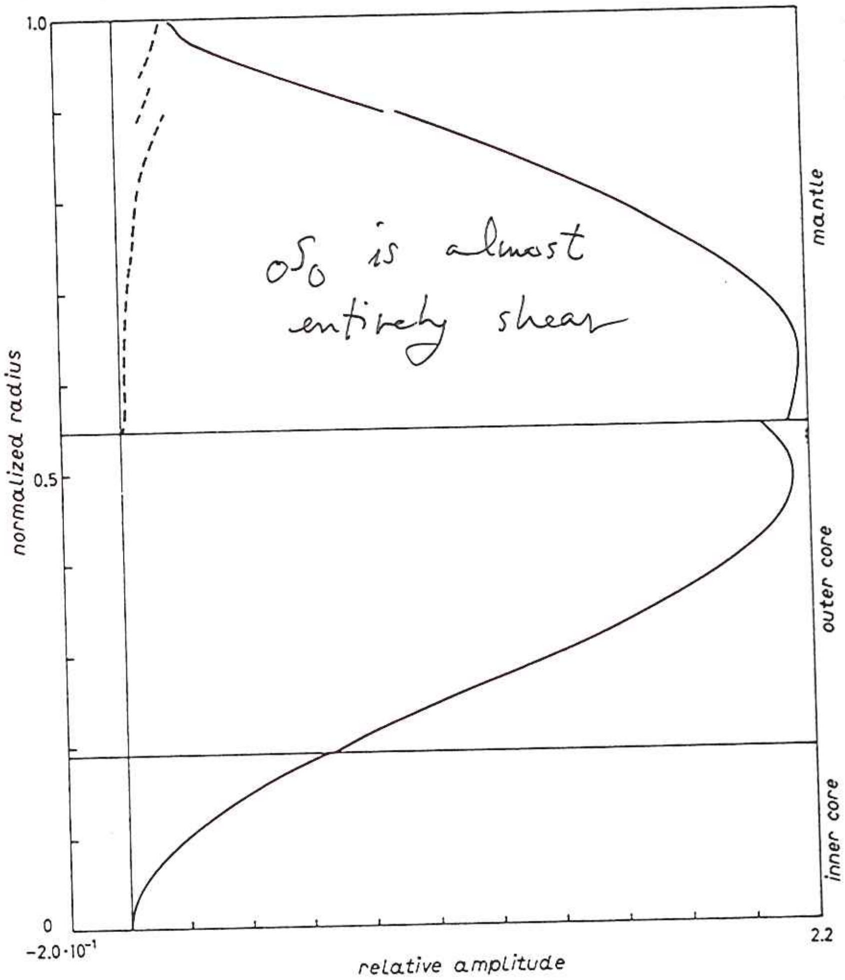
$$V_s = \frac{1}{2}\int_0^a \underbrace{\mu \left[\frac{1}{3}\left(2\partial_r U - \frac{2U}{r} + \frac{l(l+1)V}{r}\right)^2 + l(l+1)\left(\partial_r V + \frac{U}{r} - \frac{V}{r}\right)^2 + l(l+1)(l-1)(l+2)\frac{V^2}{r^2}\right] r^2}_{\text{density}} dr$$





surface wave (Rayleigh) modes are mostly shear

Fig. 12. – The mode ${}_0S_{19}$: compressional-energy density (——) and shear energy density (– – –).



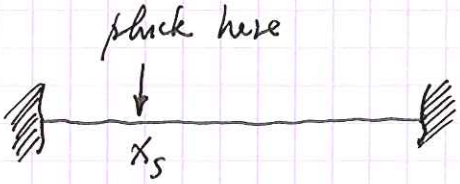
${}_0S_0$ is almost entirely shear

Fig. 13. – Compressional (——) and shear (– – –) energy densities for ${}_0S_0$.

Mode – Ray Duality

We consider next the correspondence between normal modes or standing waves and travelling body waves.

Analogy : a violin string

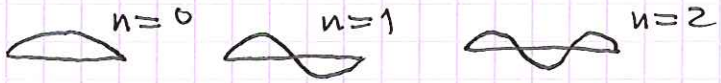


The response can be written as a sum of standing waves :

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{1}{\rho c n\pi}\right) \sin\left(\frac{n\pi x_s}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right)$$

Note : reciprocity – $x \leftrightarrow x_s$

$c = \sqrt{T/\rho}$ wave speed



ρ = density , T = tension

L = length

Or as a sum of propagating pulses

$$u(x,t) = \frac{1}{2\rho c} \sum_{j=1}^{\infty} e^{-iN_j\pi} H(t - d_j/c)$$

The distance travelled by the j^{th} pulse is d_j. Each reflection off the end gives rise to a phase change of π

$d_1 = |x - x_s|$ $\quad N_1 = 0$

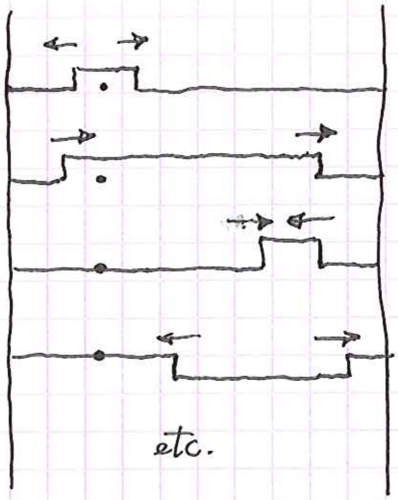
$d_2 = x + x_s$ $\quad N_2 = 1$

$d_3 = 2L - (x + x_s)$ $\quad N_3 = 1$

$d_4 = 2L - |x - x_s|$ $\quad N_4 = 2$

$d_{j+4} = 2L + d_j$ $\quad N_{j+4} = N_j + 2$

Snapshots:



just after strike at point •

after reflection off left wall

after reflection off right wall

after two pulses have passed through each other at "antipode"

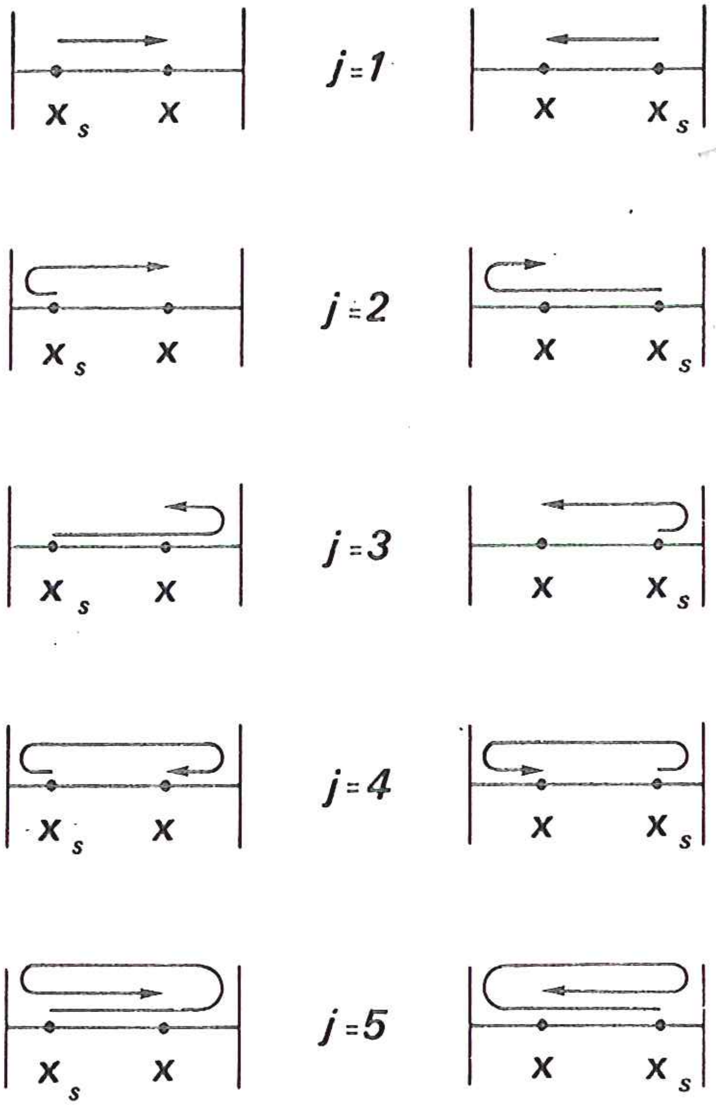
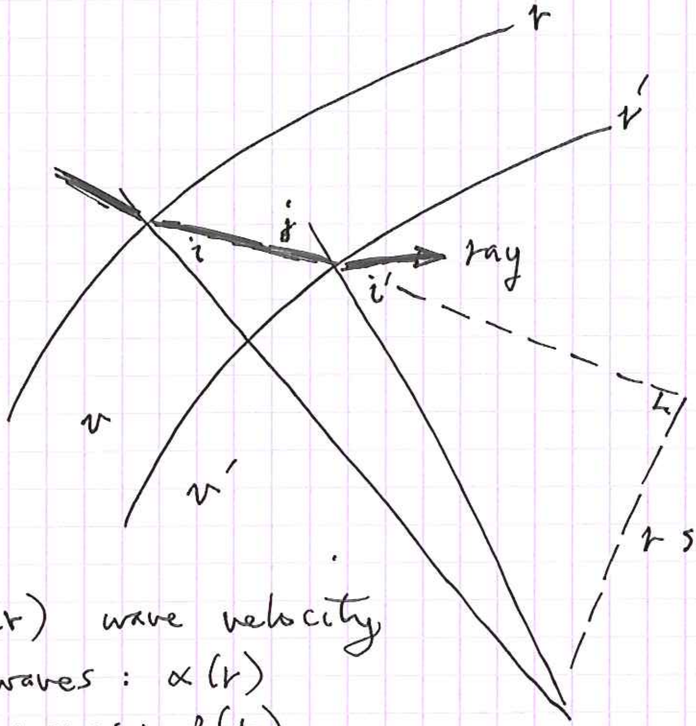


FIG. 1. The convention used for labeling waves on a string. The travel paths for the waves labeled by $j = 1$ to 5 are shown: *left*, if $x_s < x$, *right*, if $x > x_s$. In both cases, $j = 2$ is reflected once at $x = 0$ and $j = 3$ is reflected once at $x = 1$. In general, the index j does not denote the order of arrival.

We begin with a quick review of ray theory



Snell's law:

$$\frac{\sin j}{v} = \frac{\sin i'}{v'}$$

Multiply by r'

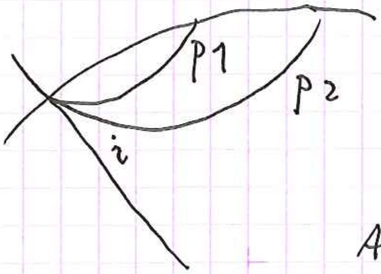
$$\frac{r' \sin j}{v} = \frac{r' \sin i'}{v'}$$

$$r \sin i = r' \sin j$$

Thus: $$\frac{r \sin i}{v} = \frac{r' \sin i'}{v'}$$

$v(r)$ wave velocity
P waves : $\alpha(r)$
S waves : $\beta(r)$

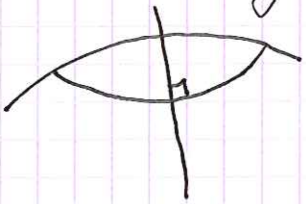
The <u>ray</u> parameter $p = \frac{r \sin i}{v}$ is <u>constant</u> along a ray



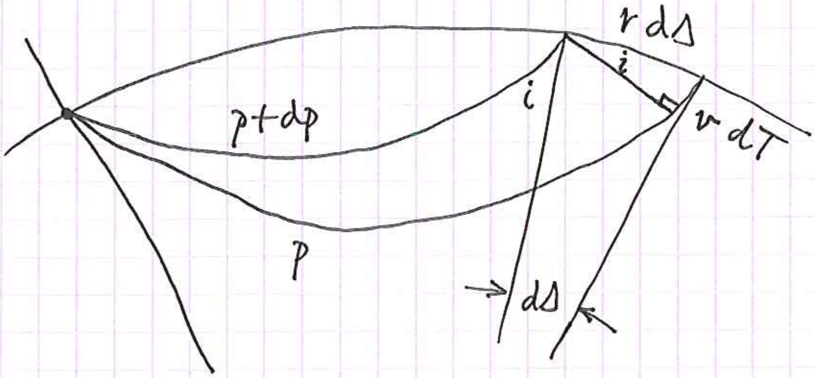
$p_2 < p_1$

units of p: seconds per radian

At bottoming point of ray $i = 90°$



$p = r_p / v(r_p)$
r_p = bottoming depth

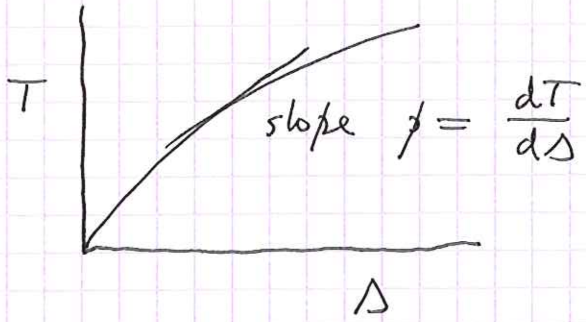


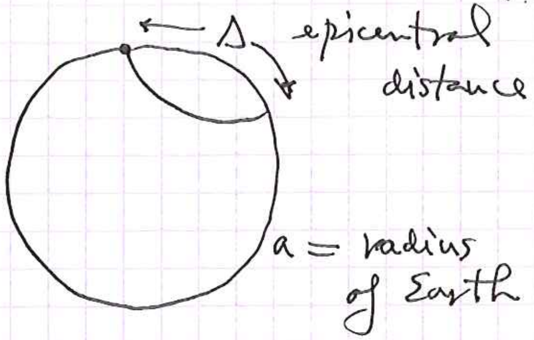
$$\sin i = \frac{v\, dT}{r\, d\Delta}$$

$$\frac{r \sin i}{v} = \frac{dT}{d\Delta}$$

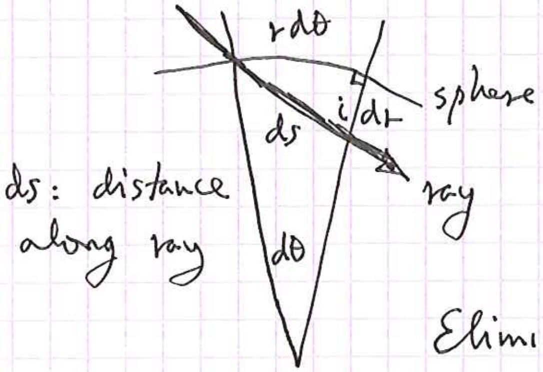
$p = \frac{dT}{d\Delta}$, slope of travel time curve

Consider 2 rays p and $p+dp$ leaving same source





The forward problem of ray theory is to calculate T versus Δ for a given velocity model v(r)



two eqns →

$$\begin{cases} ds = \frac{r}{\sin i}\, d\theta = \frac{r^2}{vp}\, d\theta \\ ds^2 = dr^2 + r^2 d\theta^2 \end{cases}$$

Eliminate dθ from these equations

$$dT = \frac{ds}{v} = \pm \frac{dr}{v^2 \sqrt{v^{-2} - p^2 r^{-2}}}$$

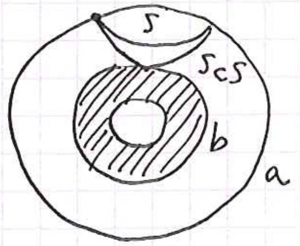
↖ sign depends on whether ray is going up or down

$$T = \int_{\text{ray}} dT$$

For a <u>turning</u> ray: P or S

$$T = 2 \int_{r_p}^{a} \frac{dr}{v^2 \sqrt{v^{-2} - p^2 r^{-2}}}$$

turning pt. → r_p



For a reflected ray: PcP or ScS

$$T = 2 \int_{b}^{a} \frac{dr}{v^2 \sqrt{v^{-2} - p^2 r^{-2}}}$$

CMB → b

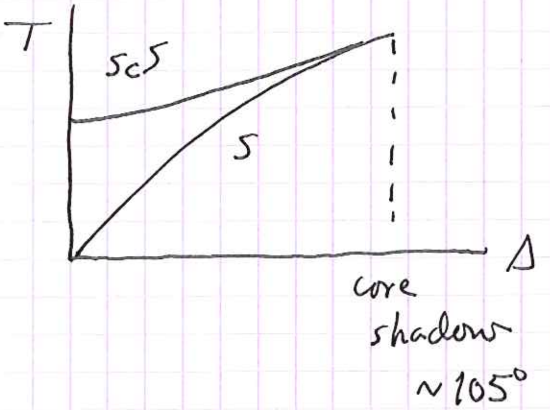
This gives T(p) for a ray of parameter P.

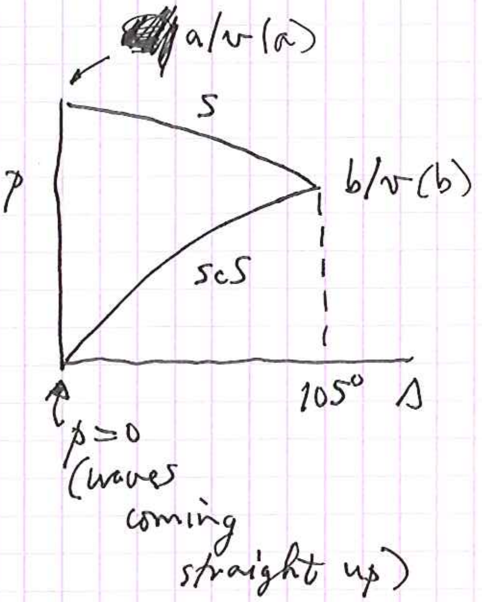
If instead eliminate ds from 2 eqns find

$$d\theta = \pm \frac{p}{r^2\sqrt{v^{-2}-p^2r^{-2}}} \qquad \Delta(p) = \int_{\text{ray}} d\Delta$$

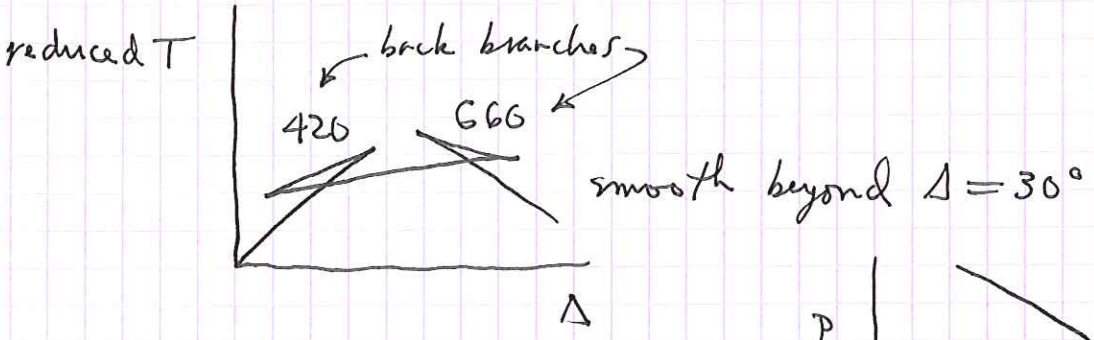
$$\Delta = 2\int_{r_p \text{ or } b}^{a} \frac{p\,dr}{r^2\sqrt{v^{-2}-p^2r^{-2}}}$$

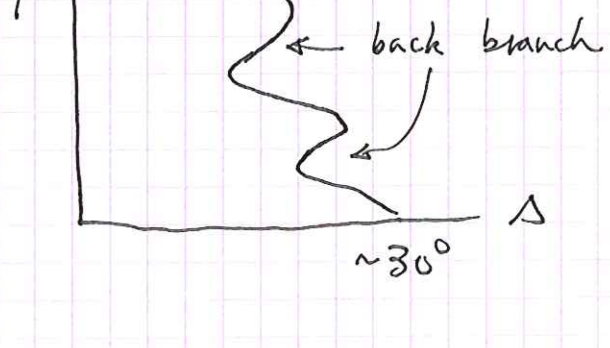
Given $T(p)$ & $\Delta(p)$ can find $T(\Delta)$



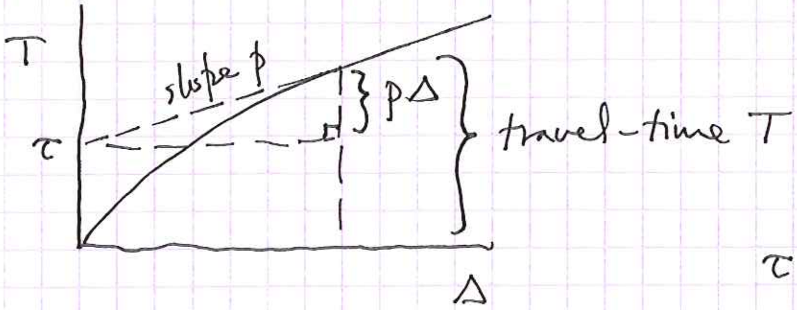


The steep velocity gradients in the upper mantle give rise to travel-time triplications.





The intercept time τ is defined by



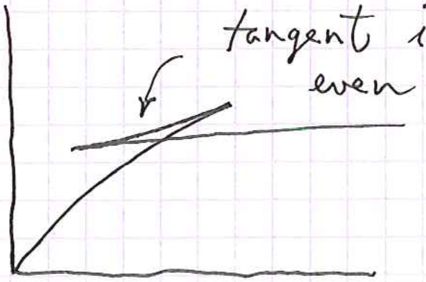
$$\tau = T - p\Delta$$

This has a number of nice properties.
By combining formulae for $T(p)$ and $\Delta(p)$ find

$$\tau(p) = 2\int_{r_p \text{ or } b}^{a} \sqrt{v^{-2} - p^2 r^{-2}}\, dr$$

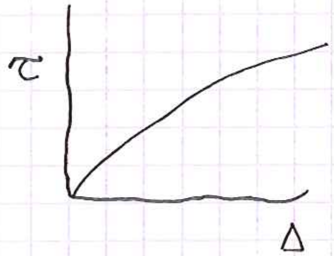
↑ no singularity

Consider a triplication:



tangent is monotonic even on back branch

Hence $\tau(\Delta)$ has no triplication



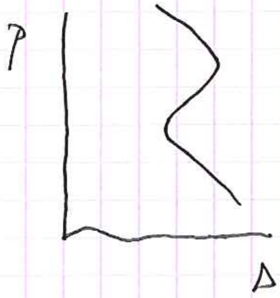
Consider $\dfrac{d\tau}{dp} = \dfrac{d}{dp}(T - p\Delta)$

$$= \frac{dT}{dp} - \Delta - p\frac{d\Delta}{dp}$$

$$\frac{d\tau}{dp} = -\Delta$$

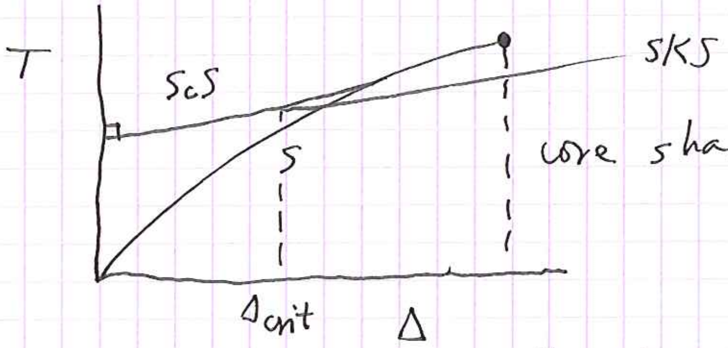
$$= \frac{dT}{dp} - \Delta - \frac{dT}{d\Delta}\frac{d\Delta}{dp} = -\Delta$$

↑ cancel ↑

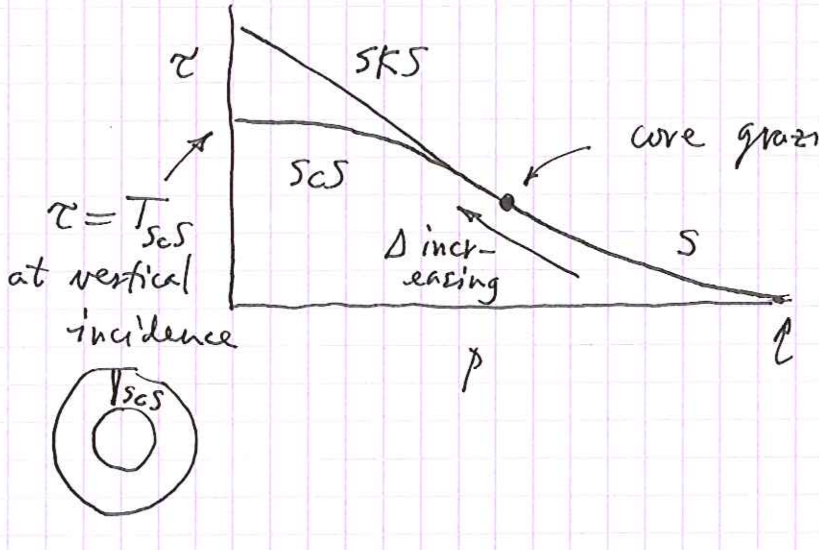


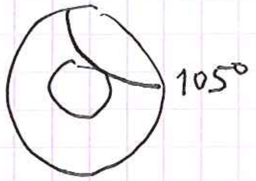
$p(\Delta)$ is triple-valued but $\Delta(p)$ is always single-valued

Thus $\tau(p)$ is monotonically decreasing and single-valued.



The $\tau(p)$ curve for SH waves looks like:

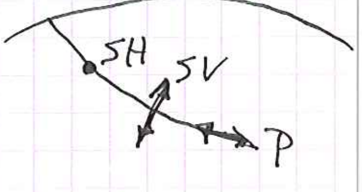




$$\frac{d\tau}{dp} = -\Delta = 0 \quad \text{at } p = \frac{a}{\beta_{const}}$$

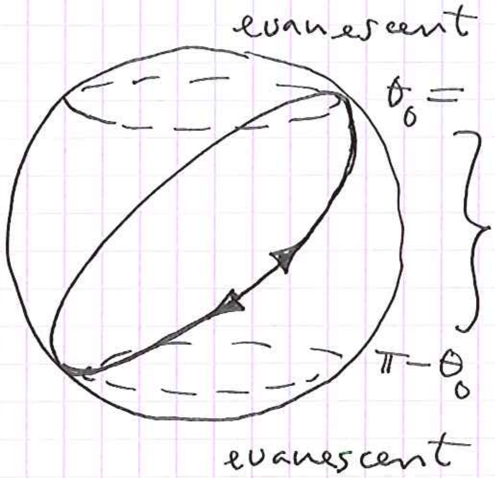
We consider now the correspondence between the toroidal modes & SH body waves

mode	body wave	surface wave
toroidal	SH	Love
spheroidal	P–SV	Rayleigh



The Jeans Relation (Sir James Jeans ~ 1920's) gives the asymptotic wavenumber of the waves equivalent to the modes ${}_nT_\ell$ or ${}_nS_\ell$.

For $\ell \gg 1$ can show that, asymptotically:



$\theta_0 = \arcsin \left| \frac{m}{\ell + 1/2} \right|$ turning colatitude

$Y_\ell^m(\theta, \phi)$ oscillatory here has character of two waves travelling in opposite directions with wavenumber

$$k = \frac{\ell + 1/2}{a}$$

Jeans relation

A special case ($m = 0$):

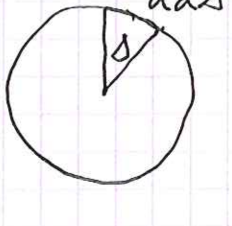
$$Y_\ell^0(\theta) \sim \frac{1}{\pi} (\sin\theta)^{-1/2} \cos\left[(\ell + \tfrac{1}{2})\theta - \frac{\pi}{4} \right] \quad \text{for } \ell \gg 1.$$

($\theta_0 = 0$ in this case)

The phase velocity is $\frac{\omega}{k} = \frac{\omega a}{\ell + 1/2}$

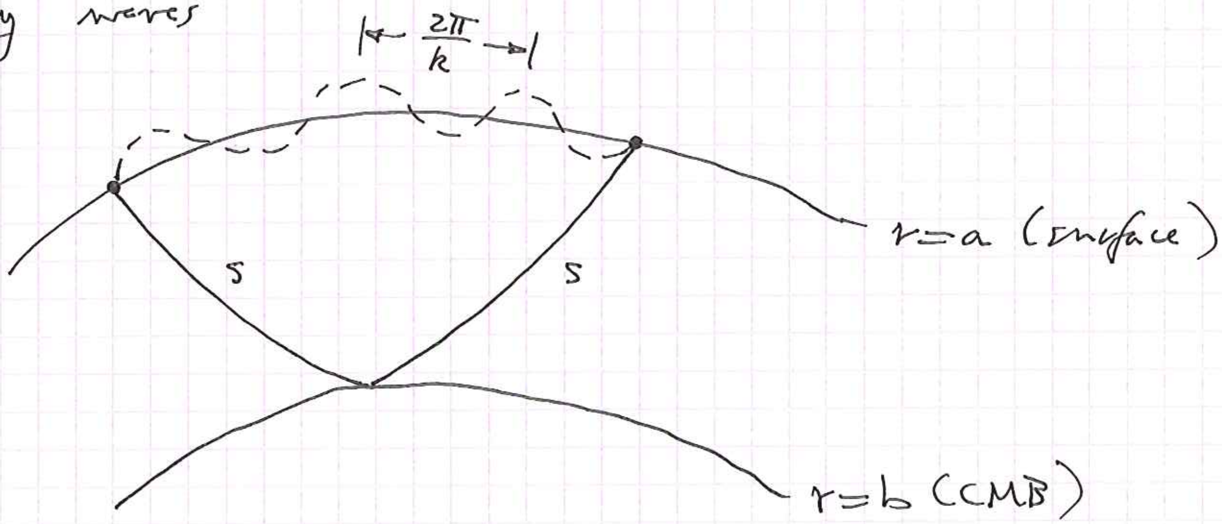
Can also write the phase velocity in terms of the ray parameter p:

$$\frac{\omega a}{\ell + 1/2} = \frac{a \, d\Delta}{dT} = \frac{a}{p}$$



Comparing, we find: $\boxed{\omega p = \ell + 1/2}$

We can obtain a simple asymptotic formula for the toroidal eigenfrequencies using a physical constructive interferance argument. We regard the mode or standing wave as the result of the constructive interference of propagating SH body waves



The (dotted - - -) phase variation along the surface is simply $\omega p \Delta$. The phase variation along the ScS ray is ωT_{ScS}. The constructive interference condition is

$$\omega T = \omega p \Delta + 2n\pi \quad (n = 0, 1, 2, \cdots)$$

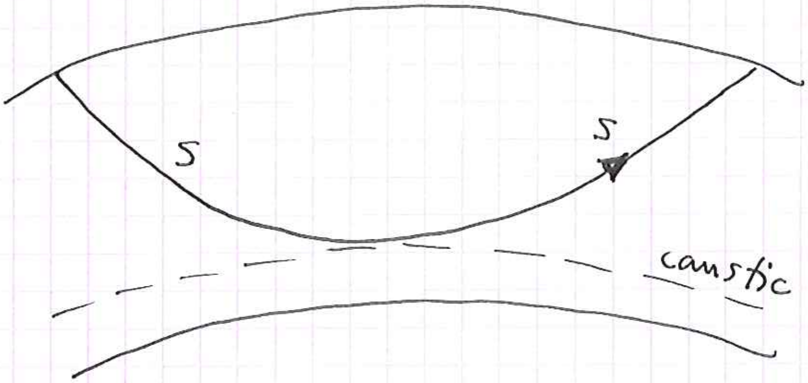
$$\boxed{\omega \tau = 2n\pi}$$

This gives 2 "semi-classical" quantization conditions to determine ${}_n\omega_\ell$:

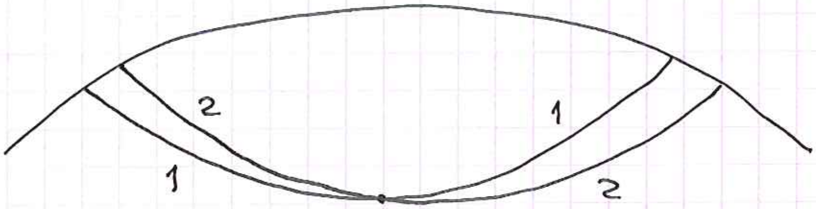
$\omega p = \ell + 1/2$ ← half-integer comes from polar (caustic) phase shift

$\omega \tau(p) = 2n\pi$

For larger p the ray *turns* rather than reflects:



The turning radius r_p is a *caustic* or envelope of the family of all rays of ray parameter p:



rays 1 and 2 cross at the caustic

As a result there is a $\pi/2$ phase shift upon turning — amplitude $\sim \frac{1}{\sqrt{\text{width}}}$ — raytube width → 0 changes sign $\sqrt{-1} = e^{i\pi/2}$

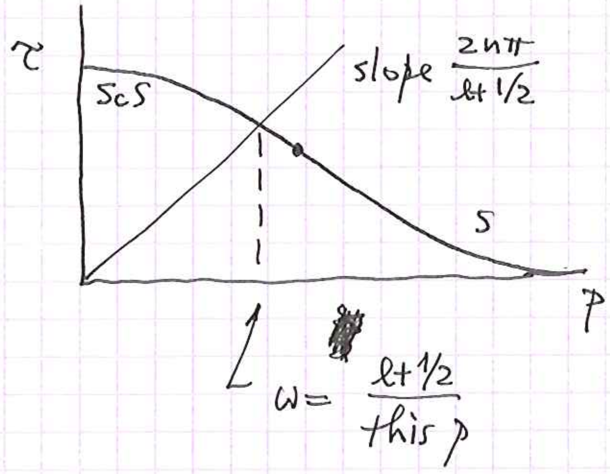
In this case we find

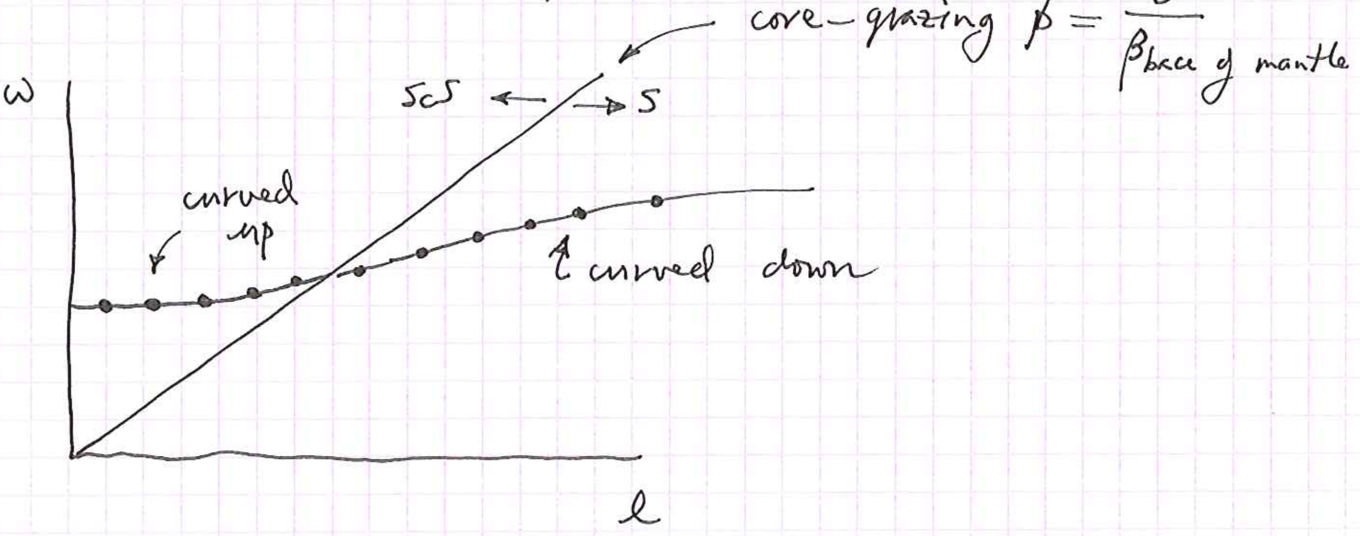
$$\underset{\substack{\uparrow\\\text{caustic}\\\text{phase shift}}}{\omega T - \pi/2} = \omega p \Delta + 2n\pi$$

$$\boxed{\omega\tau(p) = 2\pi\left(n + \underset{\substack{\uparrow\\\text{only diff}}}{\tfrac{1}{4}}\right)}$$

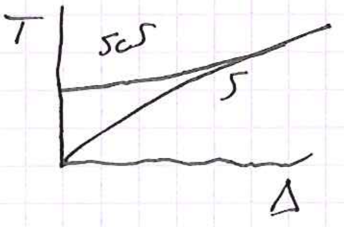
Schematically to find ω: $\tau(p) = \left(\frac{2n\pi}{l+1/2}\right)p$

or $= \left(\frac{2(n+\frac{1}{4})\pi}{l+1/2}\right)p$

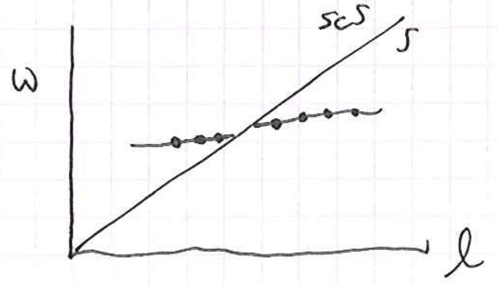


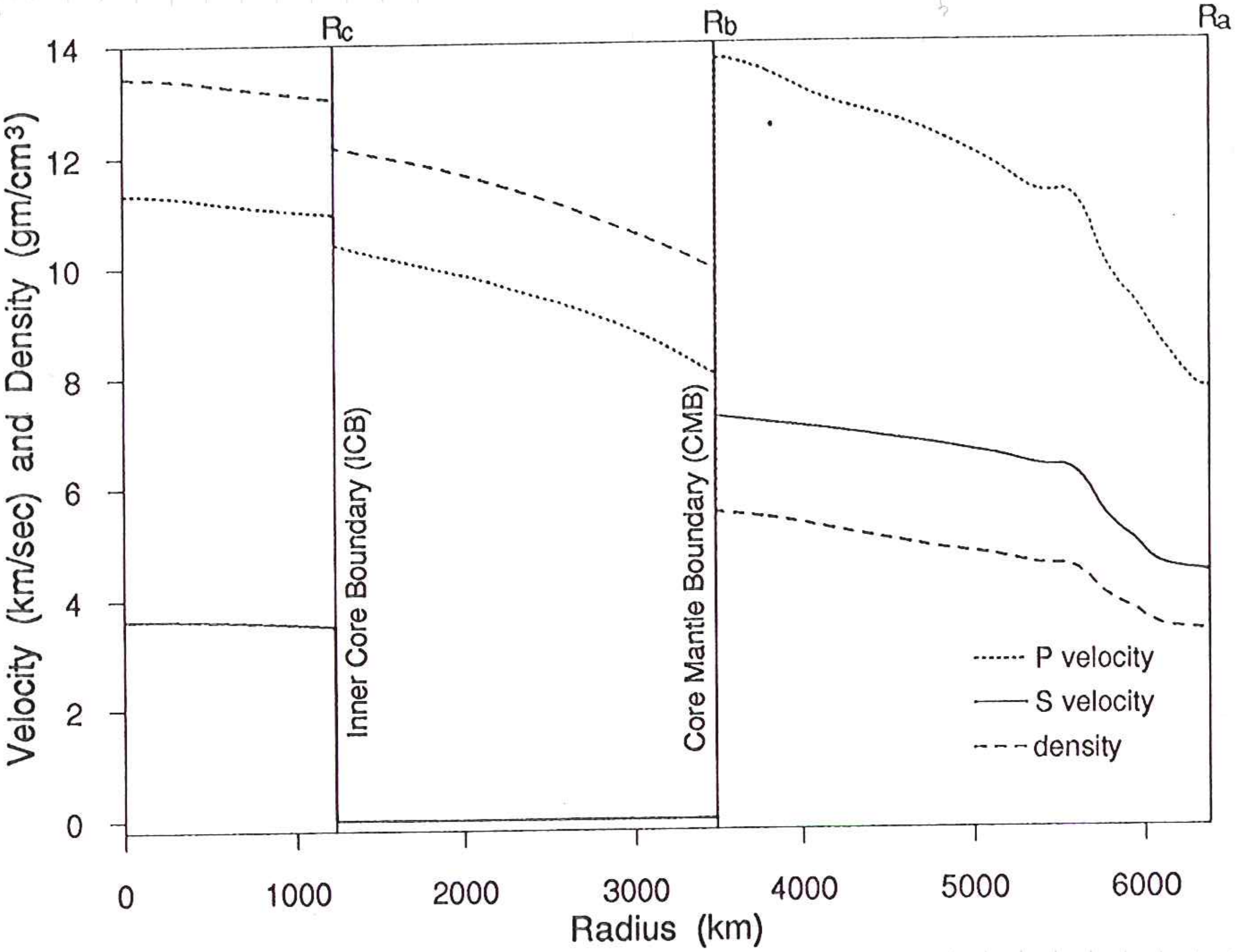


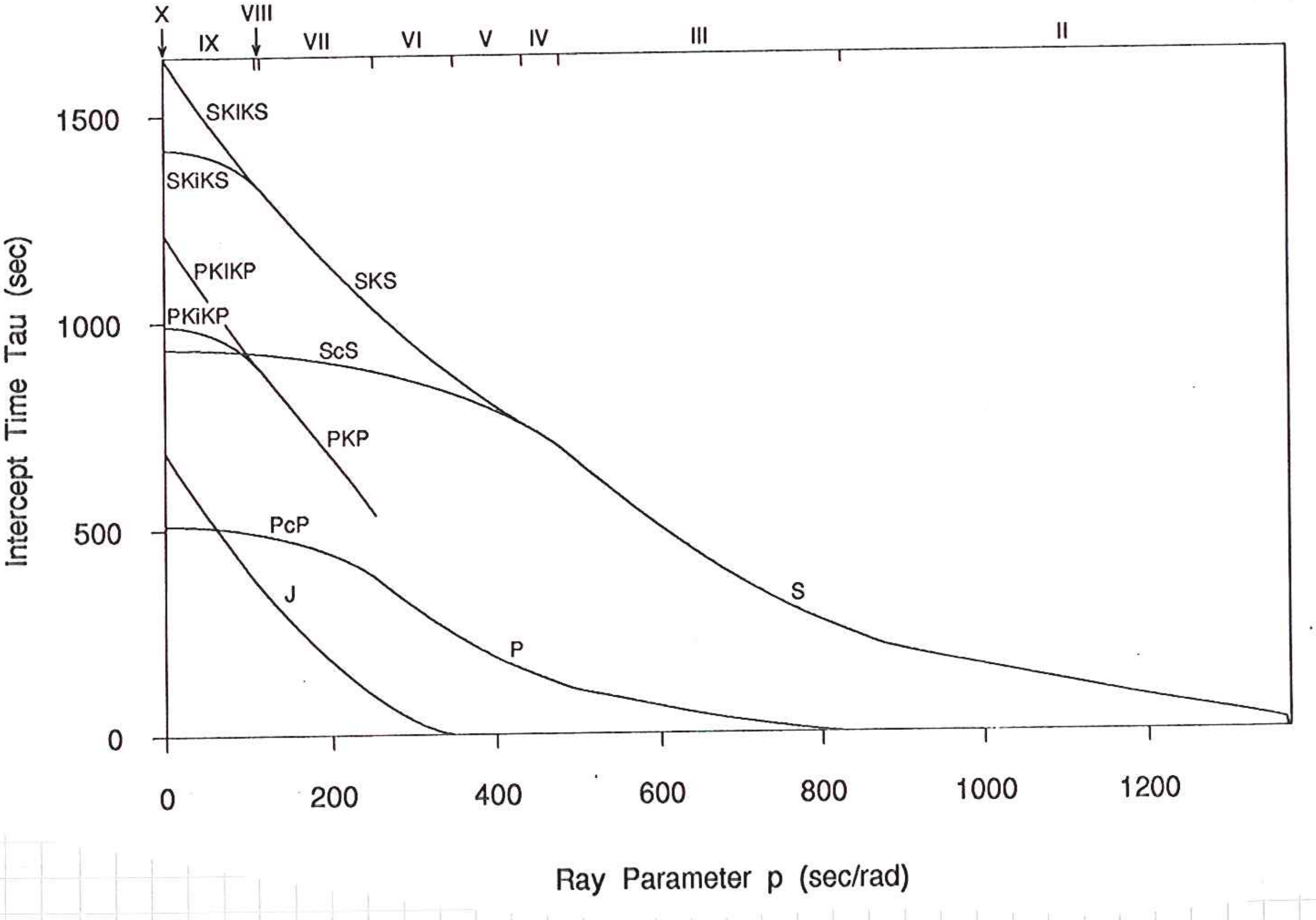
The inflection in the ${}_n\omega_l$ dispersion curves occurs at the core-grazing value of p. It's caused by the similar inflection of the $\tau(p)$ curve



The caustic phase shift gives rise to a discontinuity in the dispersion curves. This is in reality smoothed out by 'quantum' tunneling effects that are not accounted for in my theory.




Rc
Rb
Ra
Velocity (km/sec) and Density (gm/cm3)
14
12
10
8
6
4
2
0
Inner Core Boundary (ICB)
Core Mantle Boundary (CMB)
P velocity
S velocity
density
0
1000
2000
3000
4000
5000
6000
Radius (km)


X
VIII
IX
VII
VI
V
IV
III
II
1500
1000
500
0
Intercept Time Tau (sec)
SKIKS
SKiKS
PKIKP
PKiKP
SKS
ScS
PKP
PcP
J
S
P
0
200
400
600
800
1000
1200
Ray Parameter p (sec/rad)

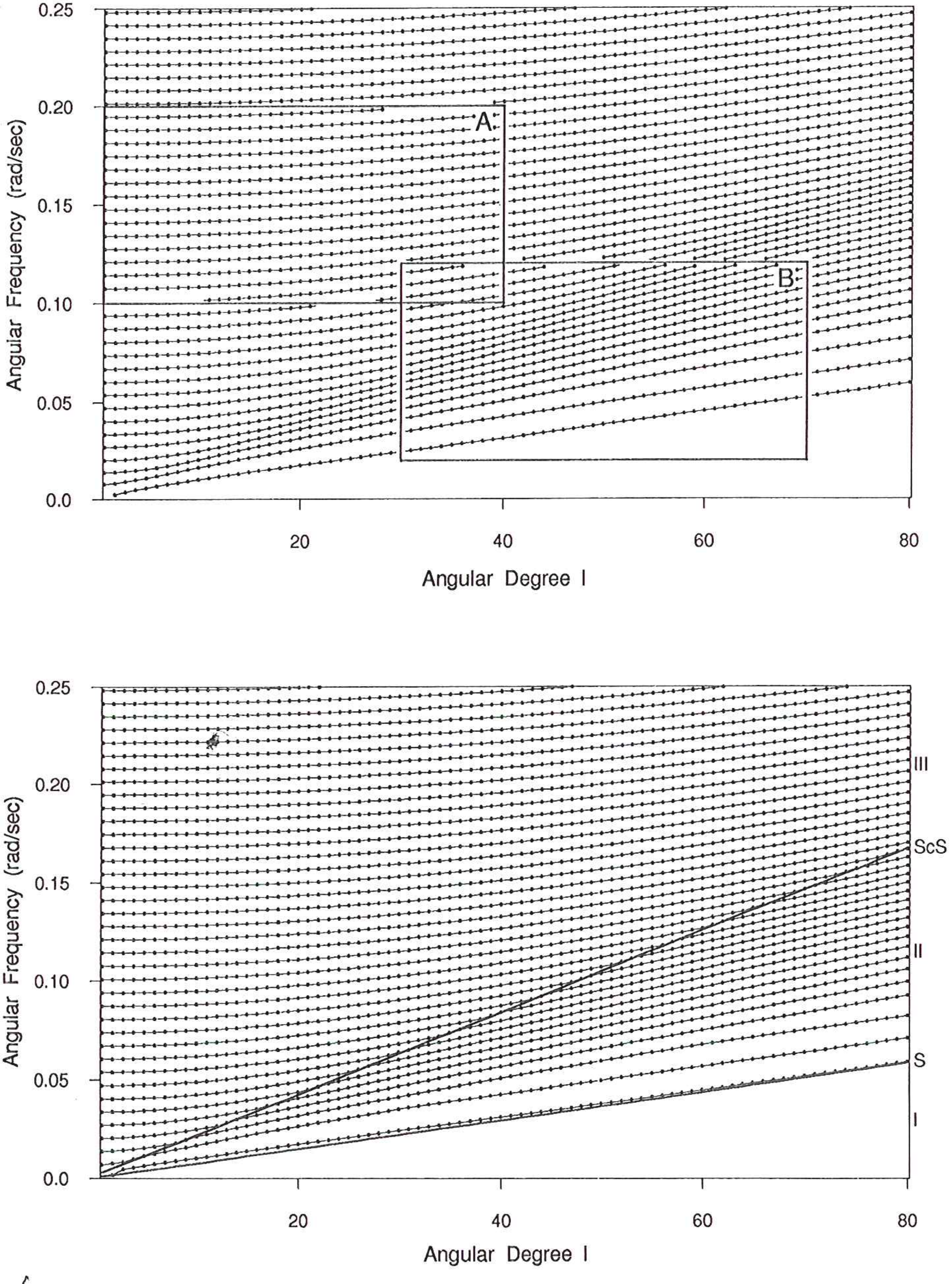
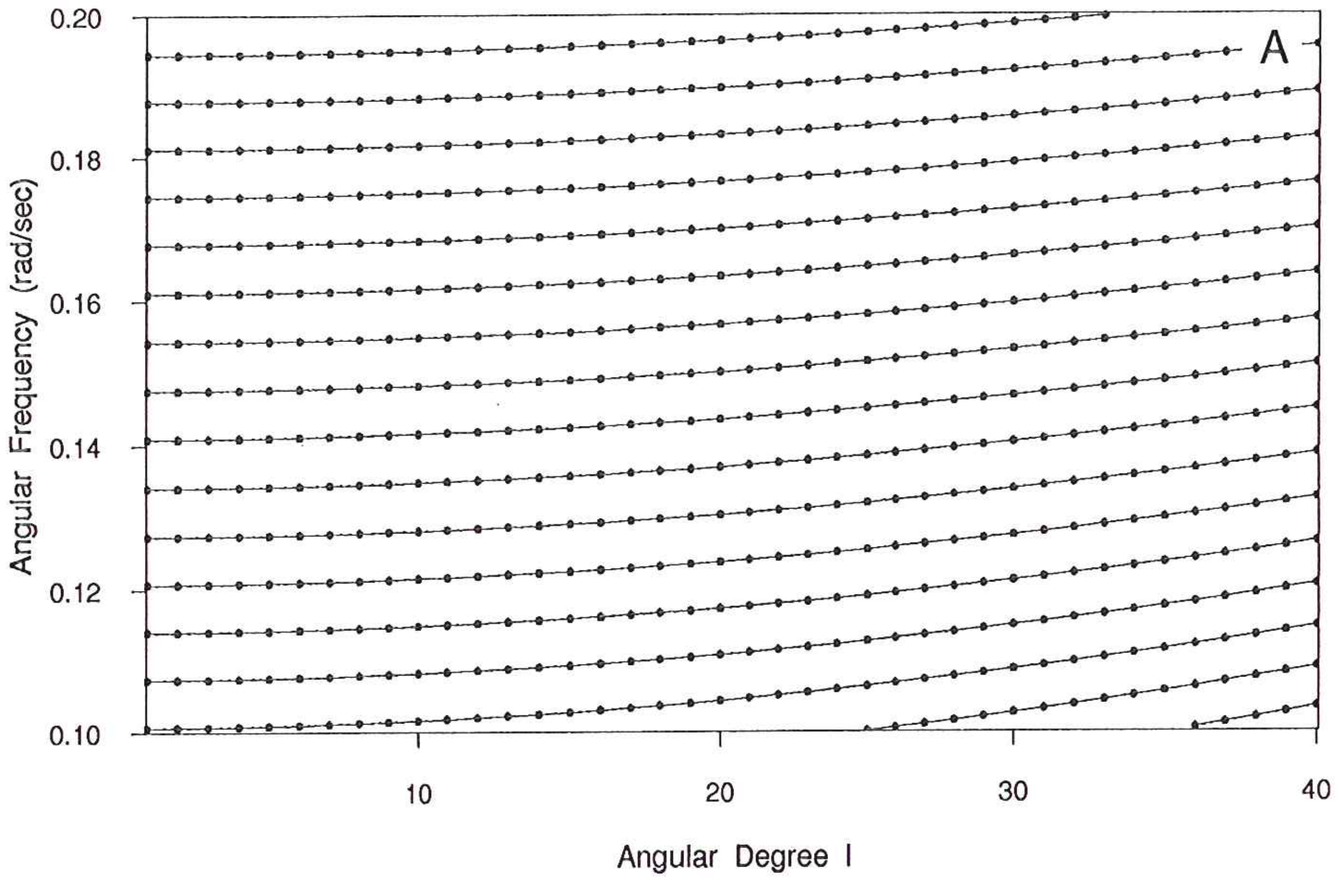
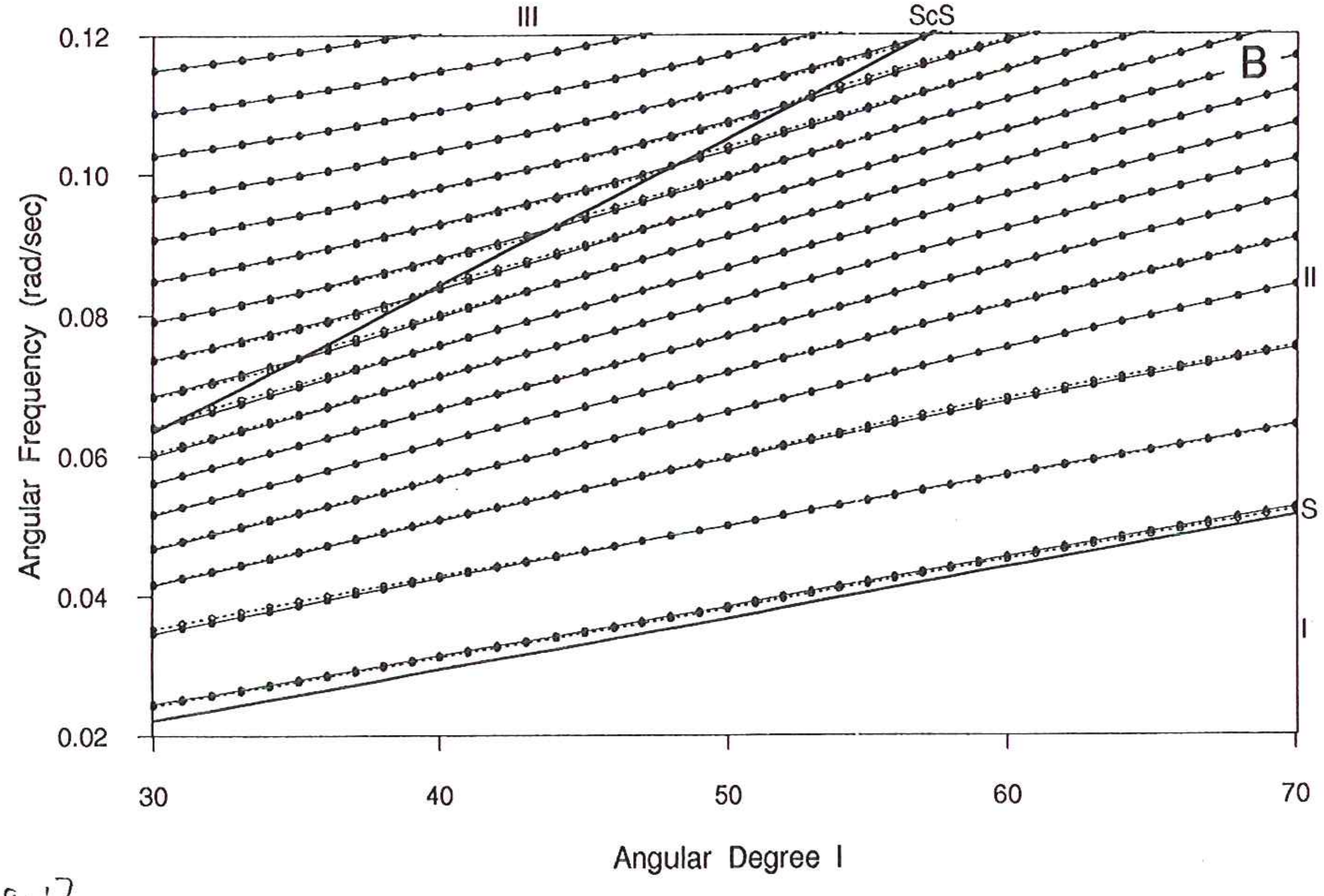

0.25
0.20
0.15
0.10
0.05
0.0
Angular Frequency (rad/sec)
A
B
20
40
60
80
Angular Degree l
0.25
0.20
0.15
0.10
0.05
0.0
Angular Frequency (rad/sec)
III
ScS
II
S
I
20
40
60
80
Angular Degree l

Fig. 6



A
Angular Frequency (rad/sec)
0.20
0.18
0.16
0.14
0.12
0.10
10
20
30
40
Angular Degree l



III
ScS
B
II
S
I
Angular Frequency (rad/sec)
0.12
0.10
0.08
0.06
0.04
0.02
30
40
50
60
70
Angular Degree l

Fig. 17

Asymptotic toroidal eigenfunctions:

In the limit $\omega \to \infty$, $p = \frac{l+1/2}{\omega}$ fixed the 2×2 toroidal system can be approximated by

$$\frac{d^2 z}{ds^2} + \omega^2 (1 - p^2\beta^2/r^2) z = 0$$

where $z = r(\rho\beta)^{1/2} W$, $s = \int_b^r \frac{dr}{\beta}$

The associated b.c. in this limit are

$$\frac{dz}{ds}(0) = \frac{dz}{ds}\left(\int_b^a \frac{dr}{\beta}\right) = 0.$$

JWKB solutions are of the form:

$$z = (1-p^2\beta^2/r^2)^{-1/4} \exp\left[\pm i \int_0^s \omega (1-p^2\beta^2/r^2)^{1/2} ds\right]$$

For ScS equivalent modes:

$$W(r) \simeq \frac{1}{r\sqrt{\rho\beta}} (1-p^2\beta^2/r^2)^{-1/4} \cos\left[\omega \int_b^r \sqrt{\beta^{-2}-p^2r^{-2}}\, dr\right]$$

or

$$W(r) = r^{-1} \mu^{-1/2} (\beta^{-2}-p^2r^{-2})^{-1/4} \cos\left[\omega \int_b^r \sqrt{\beta^{-2}-p^2r^{-2}}\, dr\right]$$

↑ satisfies b.c. at CMB

Upper b.c. is satisfied if

$$\sin\left[\omega \int_b^a \sqrt{\beta^{-2}-p^2r^{-2}}\, dr\right] = 0 \quad \text{or} \quad \sin\left(\tfrac{1}{2}\omega\tau_{ScS}\right) = 0$$

or $\omega\tau_{ScS} = 2n\pi$, as before.

Consider the normalization integral

$$\int_{\oplus} \rho s\cdot s\, dV = l(l+1)\int_a^b \rho W^2 r^2\, dr$$

$$\int_a^b \rho W^2 r^2\, dr = \frac{1}{2}\int_b^a \frac{dr}{\beta^2\sqrt{\beta^{-2}-p^2 r^{-2}}} = \frac{1}{4}T$$

from average of $\cos^2$

the travel time

The properly normalized eigenfunction is thus

$$W(r) = \frac{2}{\omega^2 p\sqrt{T_{ScS}}\, r\sqrt{\mu}}\left(\beta^{-2}-p^2 r^{-2}\right)^{-1/4}\cos\left[\omega\int_b^a\sqrt{\beta^{-2}-p^2r^{-2}}\,dr\right]$$

ScS-equivalent

or

$$W(r) = \frac{2}{\omega^2 p\sqrt{T_S}\, r\sqrt{\mu}}\left(\beta^{-2}-p^2 r^{-2}\right)^{-1/4}\cos\left[\int_{r_p}^a \omega\sqrt{\beta^{-2}-p^2r^{-2}}\,dr - \frac{\pi}{4}\right]$$

note: this is singular at turning point

turning point

this comes from solving the JWKB turning point connection problem

A uniformly valid result can be obtained using Airy functions near the turning point, or equivalently, the Langer approximation.

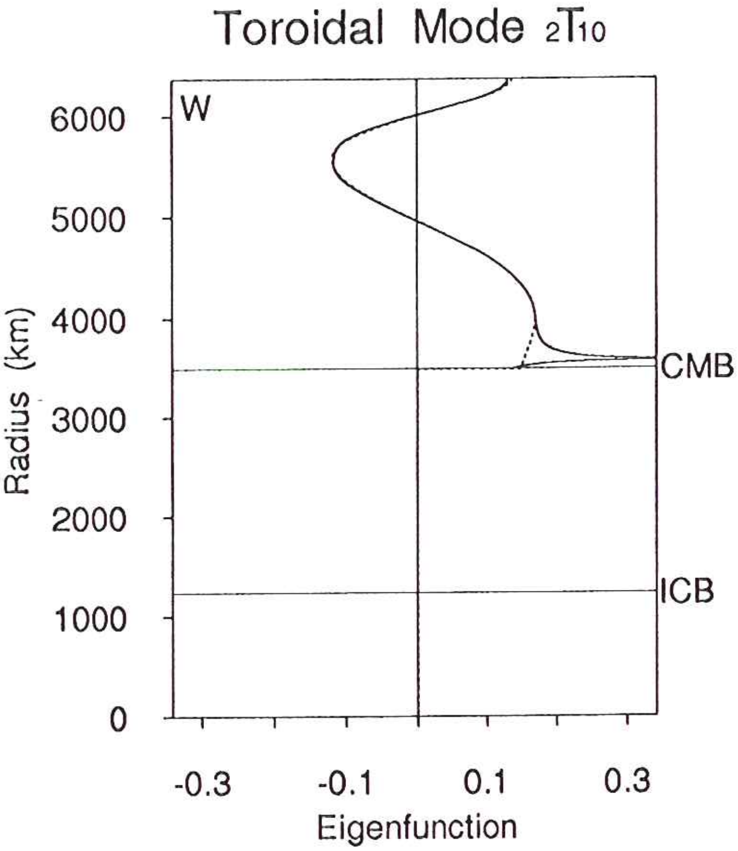


Fig. 16 (a)

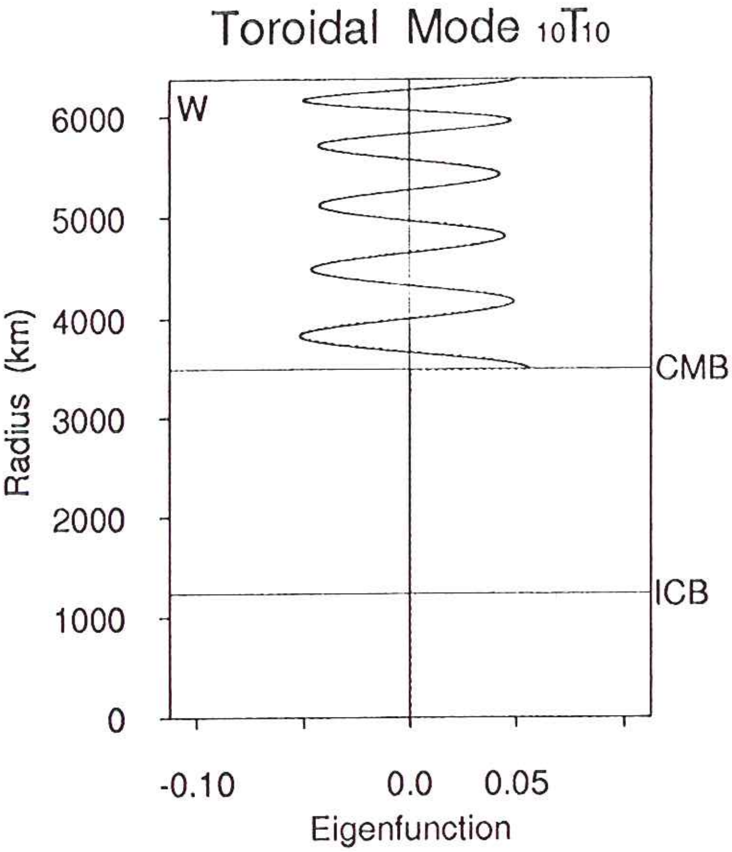


Fig. 16 (b)

Normal mode excitation on a SNREI Earth

We begin with the general formula for the acceleration response :

$$a(x,t) = \sum_k M:\varepsilon_k(x_s)\, s_k(x) \cos\omega_k t \; e^{-\omega_k t/2Q_k}$$

$$\uparrow e^{-\alpha_k t}$$

or equivalently in the Fourier transform domain:

$$a(x,\omega) = \int_0^\infty a(x,t)\, e^{-i\omega t}\, dt$$

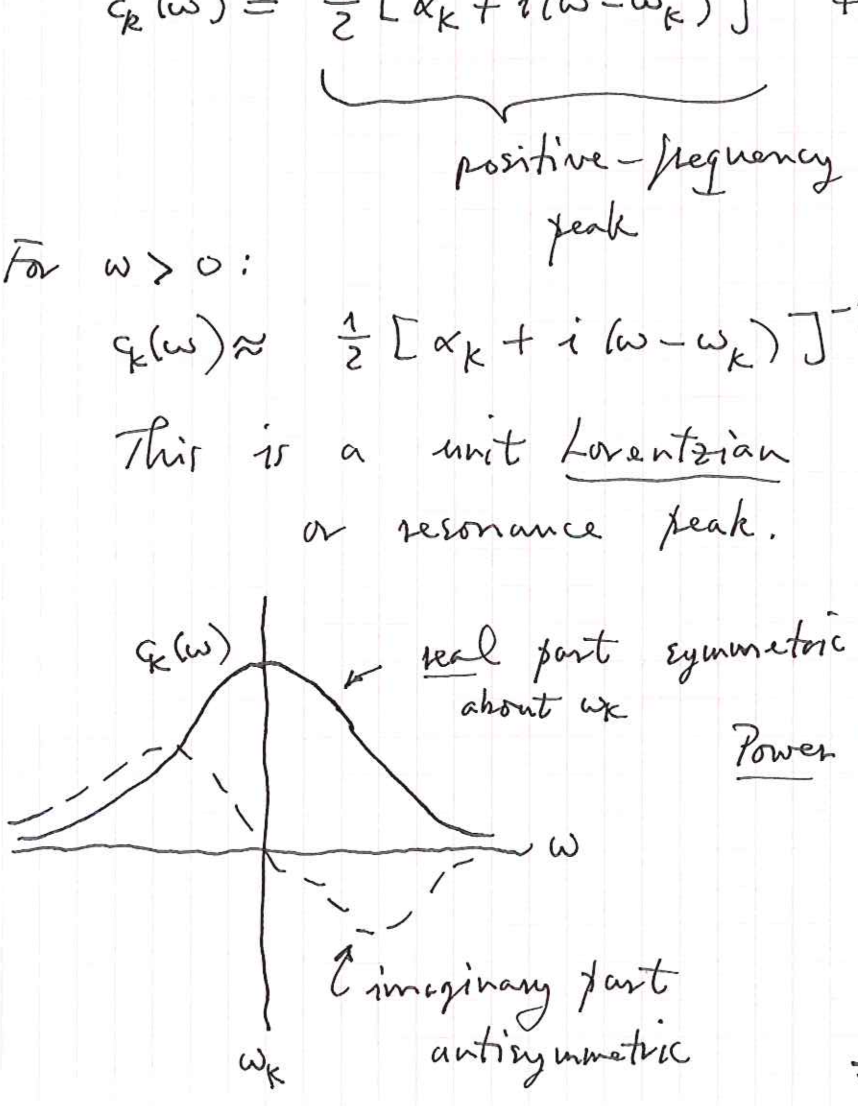
$$a(x,\omega) = \sum_k A_k(x)\, c_k(\omega)$$ where

$$c_k(\omega) = \underbrace{\tfrac{1}{2}\left[\alpha_k + i(\omega-\omega_k)\right]^{-1}}_{\text{positive-frequency peak}} + \underbrace{\tfrac{1}{2}\left[\alpha_k + i(-\omega-\omega_k)\right]^{-1}}_{\text{negative-freq peak}}$$

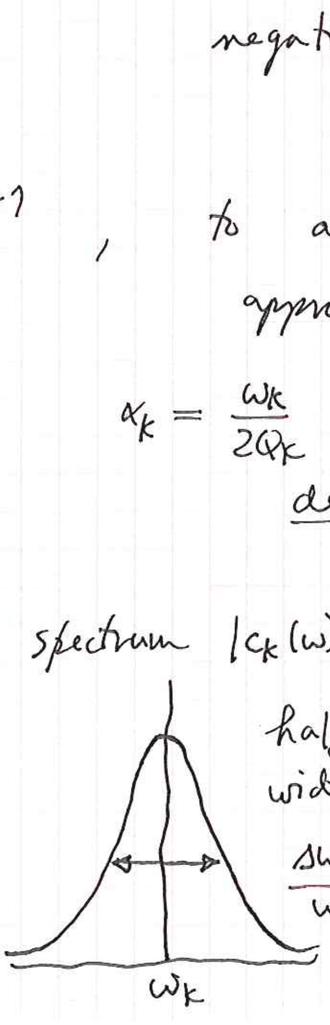
For $\omega > 0$:

$$c_k(\omega) \approx \tfrac{1}{2}\left[\alpha_k + i(\omega-\omega_k)\right]^{-1}$$, to a good approximation

This is a unit Lorentzian or resonance peak. $\alpha_k = \frac{\omega_k}{2Q_k}$ is the decay rate



Power spectrum $|c_k(\omega)|^2$:



half-power width : $\frac{\Delta\omega_k}{\omega_k} = \frac{1}{Q_k}$

The response looks like a sum of decaying cosines in the time domain, or like a sum of resonance peaks in the frequency domain.

The amplitude of each peak is:

$$A_k(\omega) = M : \varepsilon_k(x_s)\, s_k(x)$$

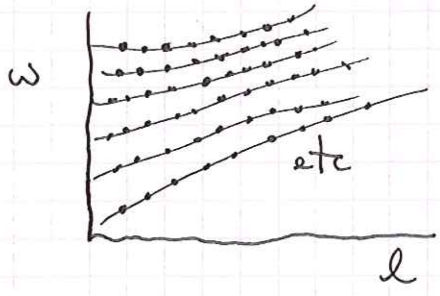
depends on quake mechanism

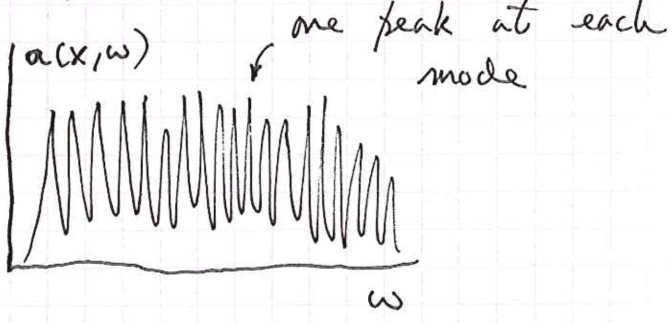
and on quake location

and on receiver location

$A_k(\omega)$ is real in this approximation — (the origin time is assumed known)

A spectrum of a single seismogram "collapses" all the dots • in the ω–l diagram over onto the frequency axis.





One is faced with a severe problem of mode identification. Some peaks are well-isolated and can be identified simply on the basis of their frequency, but many superposed peaks can't be resolved due to the line broadening caused by attenuation and finite record lengths.

To determine the response of a SNREI Earth it is convenient to use epicentral coordinates (i.e. consider the source x_s to be at the north pole)

Recall that the SNREI modes are of the form

$$s_k = {}_nW_\ell(r)\,[-\hat{r}\times\nabla_1 Y_\ell^m]\,, \quad \text{toroidal}$$

$$s_k = {}_nU_\ell(r)\,\hat{r}\,Y_\ell^m + {}_nV_\ell(r)\,\nabla_1 Y_\ell^m\,, \quad \text{spheroidal}$$

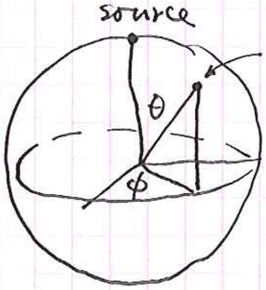
The "generic" index k stands for all of $\{S, T\}, n, \ell, m$.

spheroidal toroidal

radial overtone number

We must evaluate the number $M : \varepsilon_k(x_s)$

where $\varepsilon_k = \frac{1}{2}\left[\nabla s_k + (\nabla s_k)^T\right]$



In the limit as $\theta \to 0$, the spherical harmonics look like

$$Y_\ell^m \sim b_m\,\theta^m \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}$$

↑ go to zero like θ^m — only Y_ℓ^0 is non-zero at the pole

where $b_m = \dfrac{(-1)^m}{2^m\, m!}\sqrt{\dfrac{2\ell+1}{4\pi}}\left[\dfrac{\ell+m!}{\ell-m!}\right]^{1/2}$

To find ε_k we must differentiate Y_ℓ^m twice. As a result only $-2 \le m \le 2$ are excited.

The exact expression for $\mathscr{Y}(\hat{\mathbf{r}}, \sigma)$ on a spherical Earth has been given by Gilbert & Dziewonski (1975). Let us use $\tilde{\Sigma}_i(\Phi)$, $i = 1$–5, to denote the quantities

$$\tilde{\Sigma}_1(\Phi) = (\lambda/2\pi)^{1/2}\,[M_{zz}\,\partial_r U_s + \tfrac{1}{2}(M_{xx} + M_{yy})r_s^{-1}(2U_s - (\lambda^2 - \tfrac{1}{4})\,V_s)]$$
$$\tilde{\Sigma}_2(\Phi) = (\lambda/2\pi)^{1/2}(\lambda^2 - \tfrac{1}{4})^{1/2}(\partial_r V_s + r_s^{-1}(U_s - V_s))\,[M_{yz}\sin\Phi + M_{xz}\cos\Phi]$$
$$\tilde{\Sigma}_3(\Phi) = (\lambda/2\pi)^{1/2}(\lambda^2 - \tfrac{1}{4})^{1/2}(\lambda^2 - \tfrac{9}{4})^{1/2}r_s^{-1}\,V_s\,[M_{xy}\sin 2\Phi + \tfrac{1}{2}(M_{xx} - M_{yy})\cos 2\Phi] \qquad (53)$$
$$\tilde{\Sigma}_4(\Phi) = (\lambda/2\pi)^{1/2}(\lambda^2 - \tfrac{1}{4})^{1/2}(\partial_r W_s - r_s^{-1}W_s)\,[-M_{xz}\sin\Phi + M_{yz}\cos\Phi]$$
$$\tilde{\Sigma}_5(\Phi) = (\lambda/2\pi)^{1/2}(\lambda^2 - \tfrac{1}{4})^{1/2}(\lambda^2 - \tfrac{9}{4})^{1/2}r_s^{-1}W_s[-\tfrac{1}{2}(M_{xx} - M_{yy})\sin 2\Phi + M_{xy}\cos 2\Phi].$$

In addition, let us define $X_l^m(\Delta)$ by

$$Y_l^m(\Delta, \Phi) = X_l^m(\Delta)\exp(im\Phi). \qquad (55)$$

The exact expression for $\mathscr{Y}(\hat{\mathbf{r}}, \sigma)$ is then, for a spheroidal multiplet ${}_nS_l$,

$$\mathscr{Y}(\hat{\mathbf{r}}, \sigma) = [\tilde{\Sigma}_1(\Phi)X_l^0(\Delta) - \tilde{\Sigma}_2(\Phi)X_l^1(\Delta) + \tilde{\Sigma}_3(\Phi)X_l^2(\Delta)]c_0(\sigma), \qquad (56)$$

and, for a toroidal multiplet ${}_nT_l$,

$$\mathscr{Y}(\hat{\mathbf{r}}, \sigma) = [\tilde{\Sigma}_4(\Phi)X_l^1(\Delta) - \tilde{\Sigma}_5(\Phi)X_l^2(\Delta)]c_0(\sigma). \qquad (57)$$

Both equations (56) and (57) are valid, on a spherical Earth, for all epicentral distances Δ.

Translation: $c_0(\sigma)$ is $c_k(\omega)$ — the unit resonance peak.

$X_\ell^0(\Delta)$, $X_\ell^1(\Delta)$, $X_\ell^2(\Delta)$ are the latitudinal parts of Y_ℓ^m.

Δ is the epicentral distance

$\lambda = \ell + 1/2$ so that $\lambda^2 - \frac{1}{4} = \ell(\ell+1)$ & $\lambda^2 - \frac{9}{4} = (\ell-1)(\ell+2)$

Δ, Φ are the same as θ, ϕ

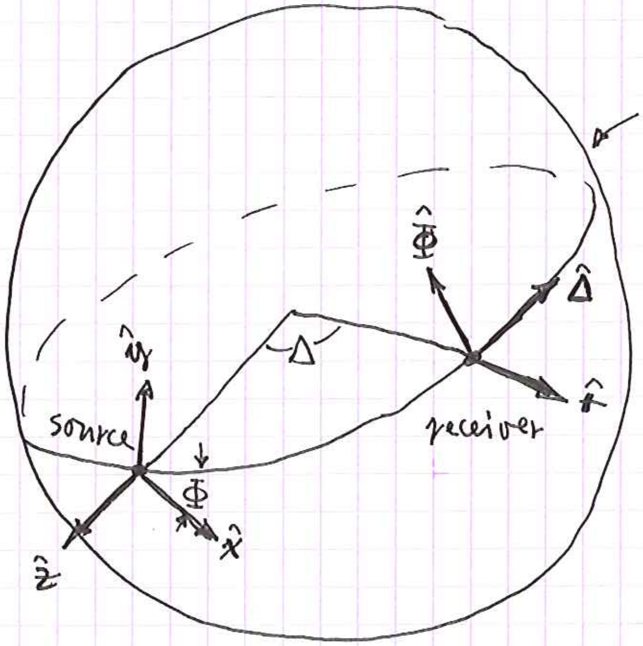
The vector polarization and amplitude of the acceleration are given by:

$$A_K(x) = W(r)\left[-\hat{r} \times \nabla_1 y(\theta,\phi)\right] : \quad \text{toroidal}$$

$$A_K(x) = \left[U(r)\,\hat{r} + V(r)\,\nabla_1\right] y(\theta,\phi) : \quad \text{spheroidal}$$

$y(\theta,\phi)$ is a kind of scalar potential

The moment-tensor components are defined in a general coordinate system:



$\hat{r}$: vertical or radial polarization

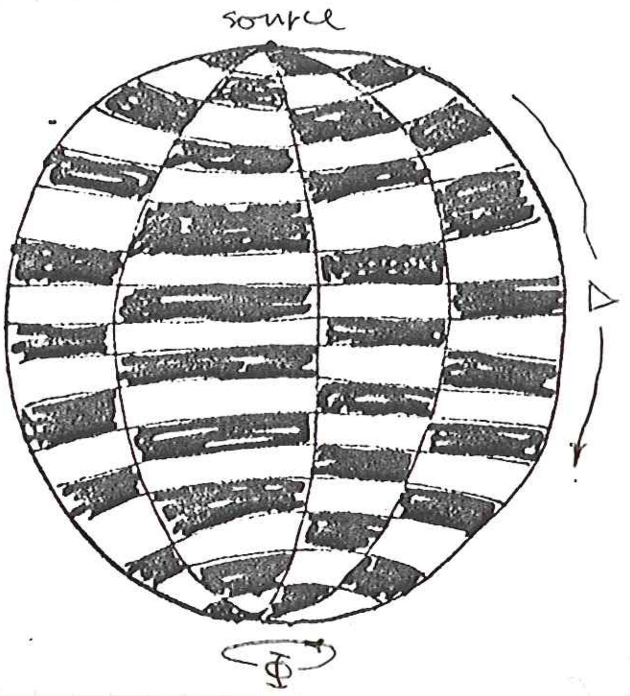
$\hat{\Delta}$: longitudinal polarization

$\hat{\Phi}$: transverse polarization

$\hat{x}$ is typically taken due <u>east</u> and $\hat{y}$ due <u>north</u>

Φ - the takeoff angle at the source is measured counterclockwise from $\hat{x}$.

The pattern of displacement associated with a given mode on the surface of the Earth looks like:



There are l nodal lines as a function of epicentral distance Δ but only 4 (or fewer — depending on the source mechanism) as a function of azimuth Φ. The phase of the oscillation is the same except for sign at every point on the Earth.

Only $-2 \leq m \leq 2$ excited

$\Rightarrow$ only 4 or fewer nodal lines in Φ.

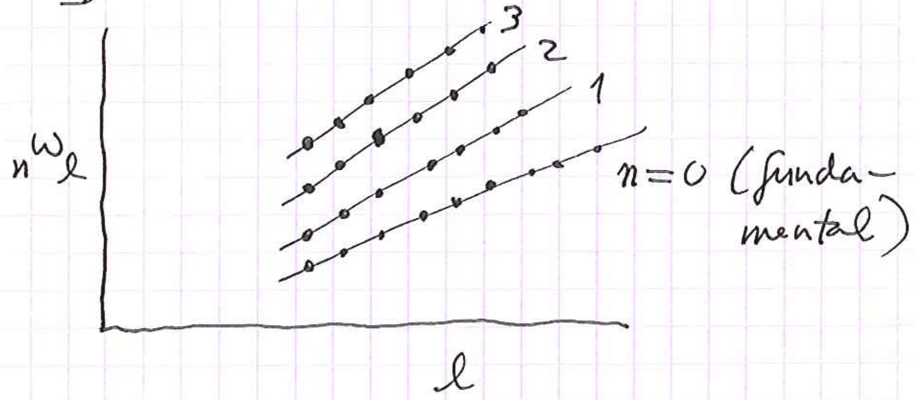
The method used to identify & isolate modes is stacking. Call the above pattern $A_k(\theta, \phi)$. Given many seismograms $a(\theta_i, \phi_i, \omega)$ one forms the stack:

$$\sum_{\substack{i \\ \text{stations}}} a(\theta_i, \phi_i, \omega)\, A_k(\theta_i, \phi_i)$$

This will reinforce the desired mode & tend to cancel all others — by this means more than 1000 modes have been identified and ω_k measured.

Decomposition into travelling surface waves

The surface-wave-equivalent modes are those having $n \ll l$:



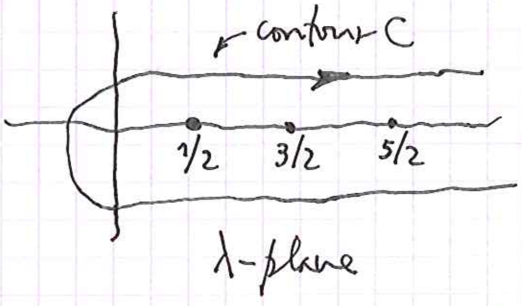
We consider these a branch at a time. We fix $n = 0, 1, 2, \cdots$ and consider the sum for $l \gg 1$.

$$a(x,\omega) = \sum_l \overset{\text{excitation amplitude}}{A_l(r,\theta,\phi)}\ \overset{\text{unit resonance peak}}{c_l(\omega)}$$

For $l \gg 1$ and $m \ll l$ we employ the asymptotic expansion:

$$X_l^m(\Delta) \sim \frac{1}{\pi}(\sin\Delta)^{-1/2}\cos\left[\left(l+\tfrac{1}{2}\right)\Delta + \frac{m\pi}{2} - \frac{\pi}{4}\right]$$

The classical technique for converting the sum over l into an integral over wavenumber k is the Watson transformation



If $f(\lambda)$ is analytic then

$$\sum_{l=0}^{\infty} f\left(l+\tfrac{1}{2}\right) = \frac{1}{2}\int_C f(\lambda)\, e^{-i\pi\lambda} \sec\pi\lambda \, d\lambda$$

Proof: $\sec \pi\lambda = \dfrac{1}{\cos\pi\lambda}$ has simple poles at

$$\lambda = 1/2,\ 3/2,\ 5/2,\ \cdots$$

Evaluate $\int_C$ by the residue theorem:

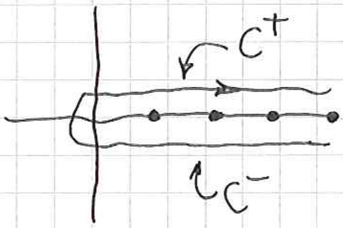
$$-2\pi i \sum \text{residues} = -\pi i \sum_{\ell} f(\ell+\tfrac{1}{2})\ \text{Res}_{\ell+1/2} \frac{e^{-i\pi\lambda}}{\cos\pi\lambda}$$

↑ because C goes around poles in cw direction

$$\text{Res}_{\ell+1/2} \frac{e^{-i\pi\lambda}}{\cos\pi\lambda} = \frac{e^{-i\pi(\ell+1/2)}}{-\pi\sin(\ell+\frac{1}{2})\pi} = i/\pi$$

So $-2\pi i \sum \text{Res} = f(\ell+\frac{1}{2})$. QED

More useful in our problem is a variant of this result known as the Poisson sum formula. The two results are equivalent despite numerous allegations to the contrary in the geophysical literature



On C^+ we can write

$$\sec\pi\lambda = 2\sum_{k=0}^{\infty} (-1)^k e^{i(2k+1)\pi\lambda} \quad *$$

Proof: $\sec\pi\lambda = \dfrac{2}{e^{i\pi\lambda}+e^{-i\pi\lambda}} = \dfrac{2e^{i\pi\lambda}}{1+e^{2i\pi\lambda}}$

$$= 2\sum_{k=0}^{\infty} (-1)^k e^{i\pi\lambda} e^{2\pi i k\lambda}$$

* Converges on C^+ where $\text{Im}\,\lambda > 0$ but not on C^- where $\text{Im}\,\lambda < 0$.

But by a similar argument:

$$\sec \pi\lambda = -2 \sum_{k=-\infty}^{-1} (-1)^k e^{i(2k+1)\pi\lambda}$$ and this converges on C^-

Combining $\int_C = \int_{C^+} + \int_{C^-}$ we get

$$\sum_{l=0}^{\infty} f(l+\tfrac{1}{2}) = \sum_{k=-\infty}^{\infty} (-1)^k \int_0^{\infty} f(\lambda)\, e^{2i\pi k\lambda}\, d\lambda ,$$ the <u>Poisson sum formula</u>

We use this result with

$$f(l+\tfrac{1}{2}) = A_{l-1/2}(r,\theta,\phi)\, c_{l-1/2}(\omega)$$

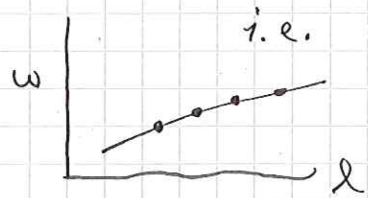
Substitute in & <u>rearrange terms</u>. Use the symmetry that U_l, V_l, W_l, ω_l, α_l are all invariant under the transformation $l+1/2 \to -l-1/2$.

Obtain the result

$$a(r,\theta,\phi,\omega) = \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} [\, a_p^+(r,\theta,\phi,\lambda)\, e^{i\lambda\Delta p}$$

$$+ \, a_p^-(r,\theta,\phi)\, e^{-i\lambda\Delta p} \,]\, c(\lambda,\omega)\, d\lambda$$

where $$c(\lambda,\omega) = \tfrac{1}{2}\,[\, \alpha(\lambda) + i(\omega-\omega(\lambda)) \,]^{-1}$$



i.e. analytic continuation ("connect the dots"

$$\Delta_p = \begin{cases} \Delta + (p-1)\pi & p \text{ odd} \\ p\pi - \Delta & p \text{ even} \end{cases}$$

where, for Rayleigh waves,

$$\mathbf{a}_p^{\pm}(\lambda, \mathbf{r}) = \frac{1}{2\pi}(\sin\theta)^{-1/2}[\hat{\mathbf{r}}U(r) \pm i\hat{\theta}\lambda V(r)]\,\mathscr{R}_1(\phi)\exp[(p-1)\pi i/2], \quad p \text{ odd}$$

$$\mathbf{a}_p^{\pm}(\lambda, \mathbf{r}) = \frac{1}{2\pi}(\sin\theta)^{-1/2}[\hat{\mathbf{r}}U(r) \mp i\hat{\theta}\lambda V(r)]\,\mathscr{R}_2(\phi)\exp(p\pi i/2), \quad p \text{ even},$$

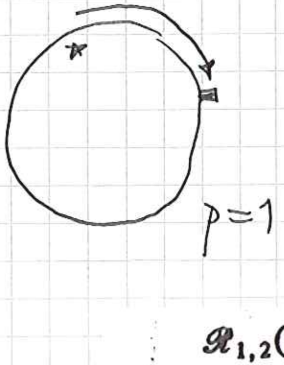
and for Love waves,

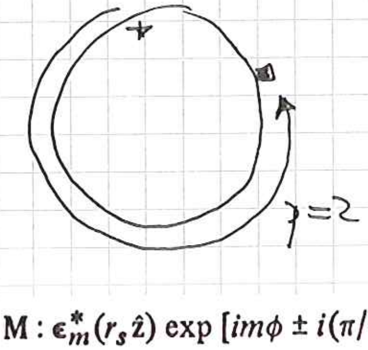
$$\mathbf{a}_p^{\pm}(\lambda, \mathbf{r}) = \frac{1}{2\pi}(\sin\theta)^{-1/2}[\pm i\hat{\phi}\lambda W(r)]\,\mathscr{R}_1(\phi)\exp[(p-1)\pi i/2], \quad p \text{ odd}$$

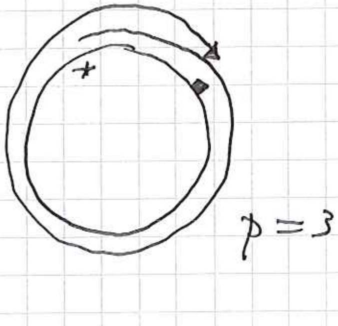
$$\mathbf{a}_p^{\pm}(\lambda, \mathbf{r}) = \frac{1}{2\pi}(\sin\theta)^{-1/2}[\mp i\hat{\phi}\lambda W(r)]\,\mathscr{R}_2(\phi)\exp(p\pi i/2), \quad p \text{ even}.$$

The $\int_{-\infty}^{\infty} d\lambda$ comes from $\int_0^{\infty} d\lambda$ in the Poisson sum formula, together with the symmetry, and the $\sum_p$ comes from the $\sum_k$ with the terms rearranged.

The quantity Δ_p is the distance travelled by the p^{th} arriving wavegroup



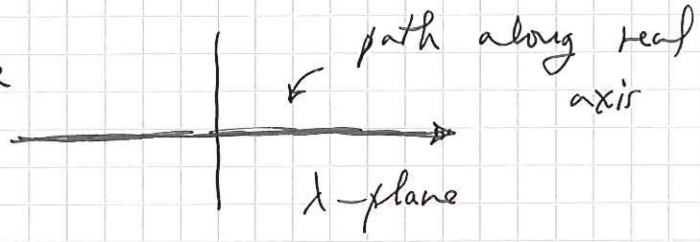




$$\mathscr{R}_{1,2}(\phi) = \sum_m \mathbf{M} : \boldsymbol{\epsilon}_m^*(r_s\hat{\mathbf{z}})\exp[im\phi \pm i(\pi/4 - m\pi/2)].$$

The quantity $\lambda = \frac{k}{a}$ is the wavenumber on the unit sphere $(ka = l + 1/2)$

The integral looks like



The factors $a_p^{\pm} e^{\pm i\lambda \Delta_p}$ are analytic. ~~the only~~ ~~singularity of the integrand is the~~

The only singularities of the integrand are those associated with $c(\lambda, \omega)$.

It has a pole at:

$$\alpha_n(\lambda) + i(\omega - \omega_n(\lambda)) = 0$$

Write $\omega_n(\lambda) \approx \omega_n(\lambda_n) + (\lambda - \lambda_n) \frac{d\omega_n}{d\lambda}(\lambda_n) + \cdots$

$$= \omega_n(\lambda_n) + (\lambda - \lambda_n)\, U_n(\lambda_n)$$

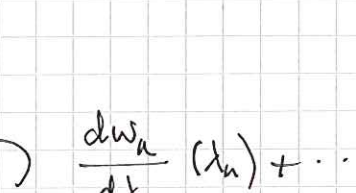
$\uparrow$ group velocity in rad/sec

Likewise to the same accuracy:

$$\alpha_n(\lambda) = \alpha_n(\lambda_n) + \cdots$$

To first-order the pole is at

$$\lambda = \lambda_n(\omega) - \frac{i\alpha_n(\omega)}{U_n(\omega)}$$



$\lambda_n(\omega)$ is the wavenumber on the unit sphere of waves of frequency ω and $\alpha_n(\omega)$ is their temporal decay rate

The wavenumber and group velocity on $r=a$ are given in terms of those on the unit sphere by

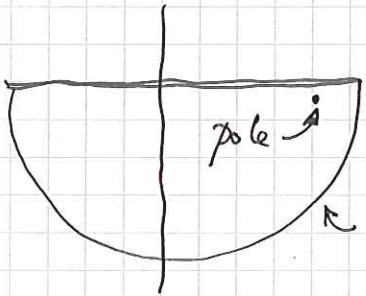
$$k_n(\omega) = \frac{\lambda_n(\omega)}{a}$$

$$U_n(\omega) = a U_n(\omega)$$

We must evaluate the integral

$$\frac{1}{2iU_n} \int_{-\infty}^{\infty} \frac{a_p^{\pm} e^{\pm i\lambda\Delta_p}}{\lambda - (\lambda_n - i\alpha_n/U_n)} d\lambda$$

For the $e^{i\lambda\Delta_p}$ integral we close the contour in the upper half plane (and get zero). For the $e^{-i\lambda\Delta_p}$ integral we close in the lower half-plane.



contribution from arc at ∞ vanishes

This gives the final result shown on the next page...

We find finally for the pth arriving group of Rayleigh waves, for frequencies $\omega > 0$, the result

$$\begin{aligned} a_p(\mathbf{r}, \omega) &\sim + \tfrac{1}{2}[\hat{\mathbf{r}} U(r) - i \hat{\boldsymbol{\theta}} \lambda_0 V(r)] (\sin\theta)^{-1/2} u_0^{-1} \exp(-\alpha_0 \Delta_p / u_0) \\ &\quad \times \mathscr{R}_1(\phi) \exp[-i\lambda_0 \Delta_p + i(p-1)\pi/2], \qquad p \text{ odd} \\ a_p(\mathbf{r}, \omega) &\sim + \tfrac{1}{2}[\hat{\mathbf{r}} U(r) + i \hat{\boldsymbol{\theta}} \lambda_0 V(r)] (\sin\theta)^{-1/2} u_0^{-1} \exp(-\alpha_0 \Delta_p / u_0) \\ &\quad \times \mathscr{R}_2(\phi) \exp[-i\lambda_0 \Delta_p + ip\pi/2], \qquad p \text{ even.} \end{aligned} \tag{174}$$

For the spectrum of the pth arriving group of Love waves, for frequencies $\omega > 0$, we obtain

$$\begin{aligned} a_p(\mathbf{r}, \omega) &\sim + \tfrac{1}{2}[i \hat{\boldsymbol{\phi}} \lambda_0 W(r)] (\sin\theta)^{-1/2} u_0^{-1} \exp(-\alpha_0 \Delta_p / u_0) \\ &\quad \times \mathscr{R}_1(\phi) \exp[-i\lambda_0 \Delta_p + i(p-1)\pi/2], \qquad p \text{ odd} \\ a_p(\mathbf{r}, \omega) &\sim + \tfrac{1}{2}[-i \hat{\boldsymbol{\phi}} \lambda_0 W(r)] (\sin\theta)^{-1/2} u_0^{-1} \exp(-\alpha_0 \Delta_p / u_0) \\ &\quad \times \mathscr{R}_2(\phi) \exp[-i\lambda_0 \Delta_p + ip\pi/2], \qquad p \text{ even.} \end{aligned} \tag{175}$$

Both equations (174) and (175) are valid, except near the poles, for waves which satisfy $\lambda_0(\omega) \gg 1$. The region of validity on Ω is $\epsilon < \theta < \pi - \epsilon$, where $\epsilon \ll \lambda_0^{-1}(\omega)$. The normalization of the functions $U(r)$, $V(r)$ and $W(r)$ is, in accordance with equation (4),

$$\int_0^a \rho_0(r) [U^2(r) + l(l+1) V^2(r)] r^2 \, dr = 1, \tag{176}$$

$$\int_0^a \rho_0(r) [l(l+1) W^2(r)] r^2 \, dr = 1. \tag{177}$$

We find, upon neglecting terms of order $\lambda_0^{-1}(\omega)$, for Rayleigh waves,

$$\begin{aligned} \mathscr{R}_{1,2}(\phi) &\sim (\lambda_0/2\pi)^{1/2} [M_{rr} \partial_r U(r_s) + \tfrac{1}{2}(M_{\theta\theta} + M_{\phi\phi}) r_s^{-1} (2U(r_s) - \lambda_0^2 V(r_s))] \exp(\pm i\pi/4) \\ &\quad - (\lambda_0/2\pi)^{1/2} \lambda_0 [\partial_r V(r_s) + r_s^{-1}(U(r_s) - V(r_s))] [M_{r\phi} \sin\phi + M_{r\theta} \cos\phi] \\ &\quad \times \exp(\mp i\pi/4) - (\lambda_0/2\pi)^{1/2} \lambda_0^2 [r_s^{-1} V(r_s)] [M_{\theta\phi} \sin 2\phi + \tfrac{1}{2}(M_{\theta\theta} - M_{\phi\phi}) \\ &\quad \times \cos 2\phi] \exp(\pm i\pi/4), \end{aligned} \tag{178}$$

and for Love waves,

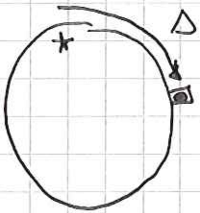
$$\begin{aligned} \mathscr{R}_{1,2}(\phi) &\sim (\lambda_0/2\pi)^{1/2} \lambda_0 [\partial_r W(r_s) - r_s^{-1} W(r_s)] [-M_{r\theta} \sin\phi + M_{r\phi} \cos\phi] \exp(\mp i\pi/4) \\ &\quad + (\lambda_0/2\pi)^{1/2} \lambda_0^2 [r_s^{-1} W(r_s)] [-\tfrac{1}{2}(M_{\theta\theta} - M_{\phi\phi}) \sin 2\phi + M_{\theta\phi} \cos 2\phi] \exp(\pm i\pi/4). \end{aligned} \tag{179}$$

Example: let's consider the first-arriving group of Love waves — it's spectrum is:

write $\lambda = ka$ and $\alpha = \omega/2Q$ and U in km/sec.

$$\boxed{\begin{aligned} a(r,\Delta,\phi,\omega) &= -\hat{\phi}\,(ka)\; W(r)\,(\sin\Delta)^{-1/2} \\ &\quad U^{-1}(\omega)\; e^{-\omega a\Delta/2QU}\;\left[-\tfrac{1}{2} i\, R_1(\phi)\right] e^{-ika\Delta} \end{aligned}}$$

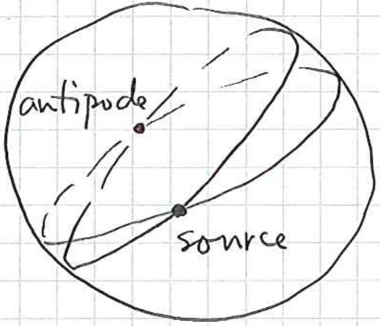
$a\Delta$ is the distance travelled in kilometers



Particle motion is transverse

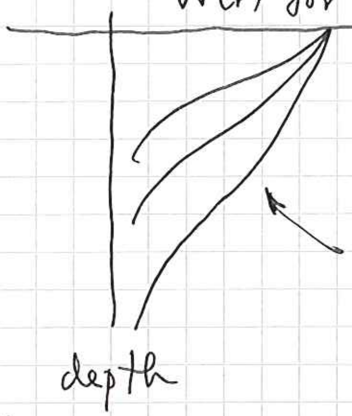
The factor $(\sin\Delta)^{-1/2}$ is the geometrical spreading in 2-d on a sphere. Signal weakest at $\Delta = 90°$ with a focus at the antipode.

raytube width $\sim \sin\Delta$



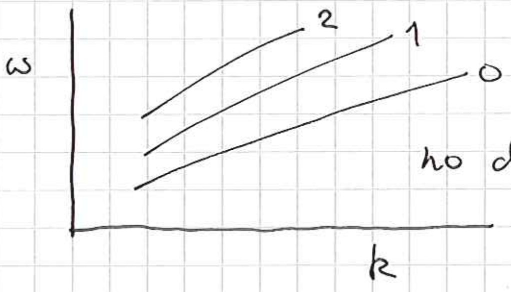
The depth dependence is that of the Love wave eigenfunction $W(r)$

$W(r)$ for $n=0$ (fundamental mode)



increasingly trapped to ⊕'s surface with increasing frequency ω and wavenumber k.

The term $e^{-ika\Delta}$ is the phase delay due to propagation. Can write in form $e^{-i\omega a\Delta/c(\omega)}$ where $c(\omega) = \frac{\omega}{k(\omega)}$ is the phase velocity of the wave. We now think of the ω-l diagram as an ω-k diagram



$c = \frac{\omega}{k}$

no dots — continuous curves

The factor $U^{-1}(\omega)$ partly, but not entirely, accounts for the dominance of frequencies corresponding to group velocity maxima — they are preferentially excited.

The term $e^{-\omega a\Delta/2QU}$ represents the physical attenuation of the wave due to anelasticity. The quality factor $Q^{-1}(\omega)$ is related to Q_K^{-1} and Q_μ^{-1} in the Earth as we have seen.

Note the appearance of the group velocity

$e^{-\frac{\omega a \Delta}{2QU}}$ not $e^{-\frac{\omega a \Delta}{2Qc}}$

This is because energy propagates with the group velocity.

Equation (7.92) can be easily understood if we measure the attenuation of dispersed waves using the stationary-phase approximation (7.18). At a given x and t, the frequency ω given by $x/t = U(\omega)$ dominates the record. Since the wave with frequency ω has existed in the medium over the time period $t = x/(U(\omega))$, it must have been attenuated by a factor

$$\exp\left[\frac{-\omega t}{2\ temporal\ Q(\omega)}\right] = \exp\left[\frac{-\omega x}{2U(\omega)\ temporal\ Q(\omega)}\right]. \tag{7.93}$$

Since, by definition, this is equal to

$$\exp\left[\frac{-\omega x}{2c(\omega)\ spatial\ Q(\omega)}\right],$$

we obtain (7.92).

Finally the term $-\frac{i}{2}\mathcal{R}_1(\phi)$ represents the Love-wave radiation pattern:

depends on strain at source

$$-\frac{i}{2}\mathcal{R}_1(\phi) = \frac{1}{2}\left(\frac{ka}{2\pi}\right)^{1/2} ka\,[\partial_r W_s - r_s^{-1} W_s]$$

$$[M_{xz}\sin\phi - M_{yz}\cos\phi]\,e^{i\pi/4}$$

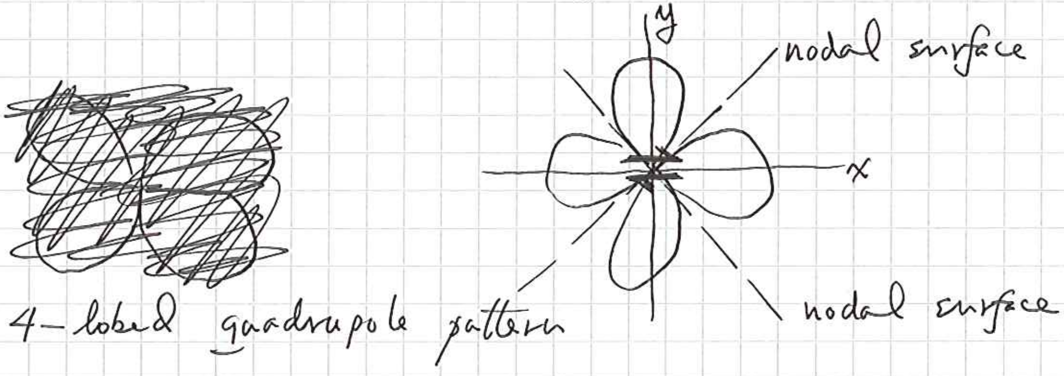
$$+\frac{1}{2}\left(\frac{ka}{2\pi}\right)^{1/2} k^2a^2\,[r_s^{-1}W_s]\left[-\frac{1}{2}(M_{xx}-M_{yy})\sin 2\phi\right.$$

$$\left.+\,M_{xy}\cos 2\phi\right]e^{-i\pi/4}$$

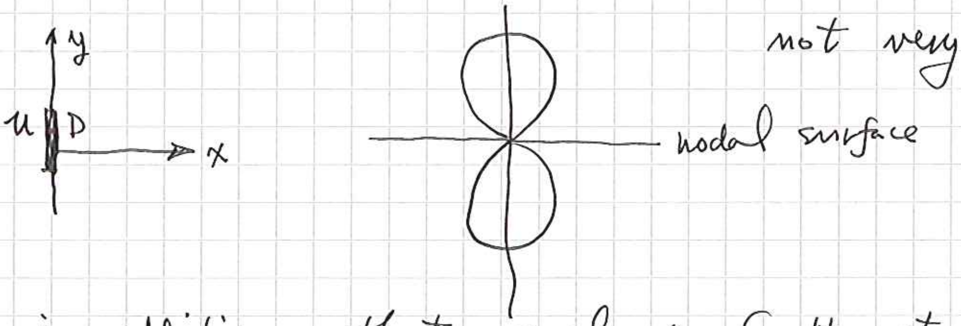
The dependence only on $\sin\phi$, $\cos\phi$, $\sin 2\phi$, $\cos 2\phi$ stems from the $-2 \leq m \leq 2$ dependence

The phase shifts $e^{\pm i\pi/4}$ "leaving" the source are important if one wishes to use phase information to invert for the mechanism M. This is, however, difficult because the phase delay due to propagation must be known and it is path-dependent. A common expedient is to use only amplitude information (esp. for smaller events)

Example – strike-slip fault ($M_{xy} \neq 0$)

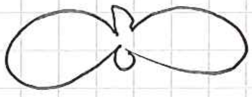


Example – vertical dip-slip (normal) fault ($M_{xz} \neq 0$)



not very intuitive

Note in addition that such a fault at the surface does not excite Love (or Rayleigh) waves (since $\partial_r W_s - r^{-1} W_s = 0$ at surface)

Combinations of the above two sources yield patterns like  or , etc.

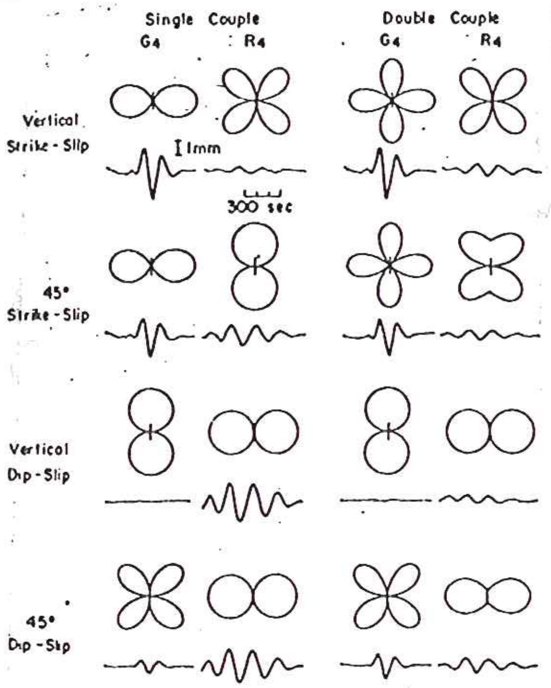


Fig. 7. Radiation pattern (maximum trace amplitude) and relative excitation of Love and Rayleigh waves for eight fundamental source geometries. The wave forms in the loop direction are shown.

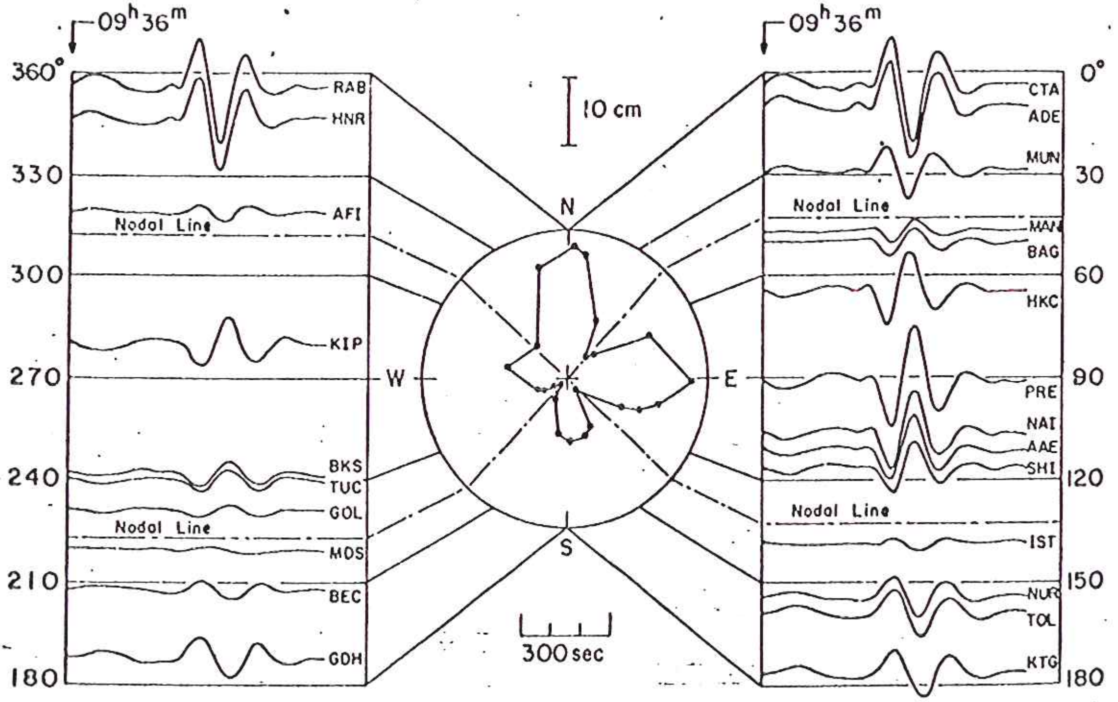


Fig. 12. Synthetic seismograms of Love waves (*G*4). The moment is 7 × 10^{28} dyne-cm. (For other parameters, see the caption for Figure 11).

LONG-PERIOD MECHANISM OF THE EUREKA, CA, EARTHQUAKE

THORNE LAY, JEFFREY W. GIVEN, AND HIROO KANAMORI

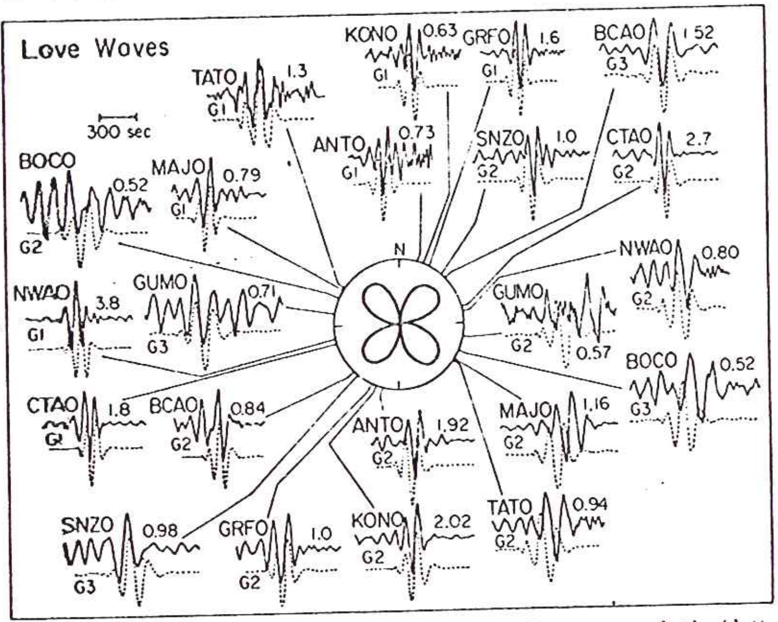
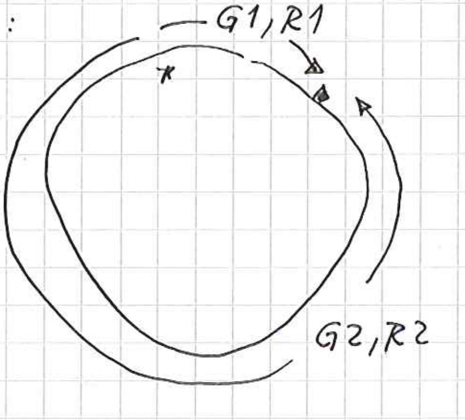


FIG. 4. The Love wave radiation pattern and comparison between synthetic (dotted lines) and observed seismograms. The fault parameters used in the synthetic calculation are listed in Table 9, and a source process time of 30 sec was assumed. The observed seismograms and synthetics for GUMO G_2, GUMO G_3, BOCO G_2, BOCO G_3, TATO G_2, TATO G_1, MAJO G_2, and MAJO G_1 were filtered with a Gaussian band-pass filter between 120 and 1500 sec. The rest of the data and synthetics were band-pass filtered between 80 and 1500 sec. The number above each observed record is the moment (in units of 10^{27} dyne-cm) that would be obtained using that record alone.

For higher orbits $p > 1$ the waves continue to be delayed $e^{-ika\Delta_p}$ and attenuated $e^{-\omega a\Delta_p/2\pi Q}$

Notation :



G1, G2 etc. for Love waves (G for Gutenburg)

R1, R2 etc. for Rayleigh waves

Can see up to R11 – R12 after a large event (see e.g. the cover of Aki & Richards)

The terms $e^{ip\pi/2}$ and $e^{i(p-1)\pi/2}$ represent the polar phase shift due to the passage through the focal points (degenerate caustics) at the source and antipode.

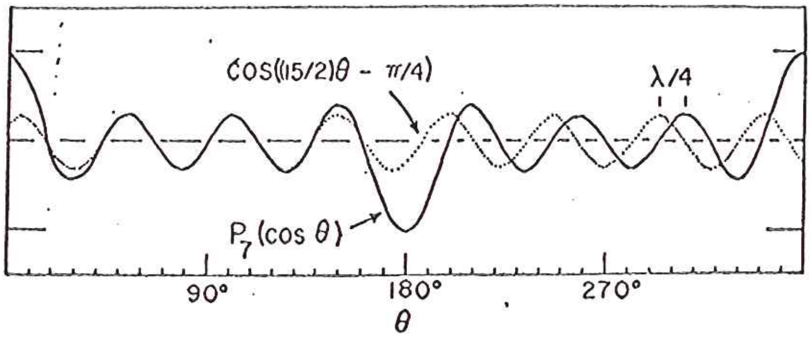


FIG. 1. A comparison of the phase of P_7 with that of its asymptotic representative.

Now, consider Rayleigh waves — for $p = 1$ we have

$$a_1(r,\theta,\phi,\omega) = +\tfrac{1}{2}\left[\hat{r}\,U(r) - ika\,\hat{\Delta}\,V(r)\right] (\sin\Delta)^{-1/2}\, U^{-1}(\omega)\, e^{-\omega a\Delta/2Q(\omega)U(\omega)}\, R_1(\phi,\omega)\, e^{-ik(\omega)\Delta}$$

The interpretation of all the factors is the same — now the motion is radial plus longitudinal.

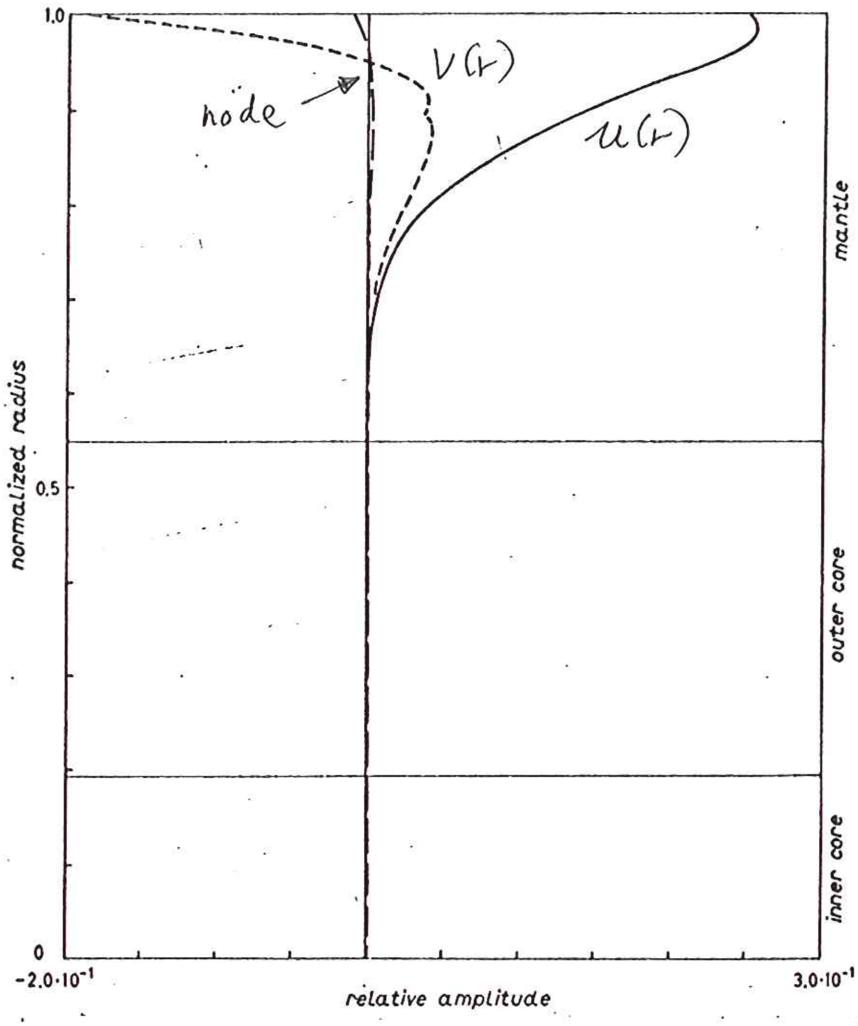
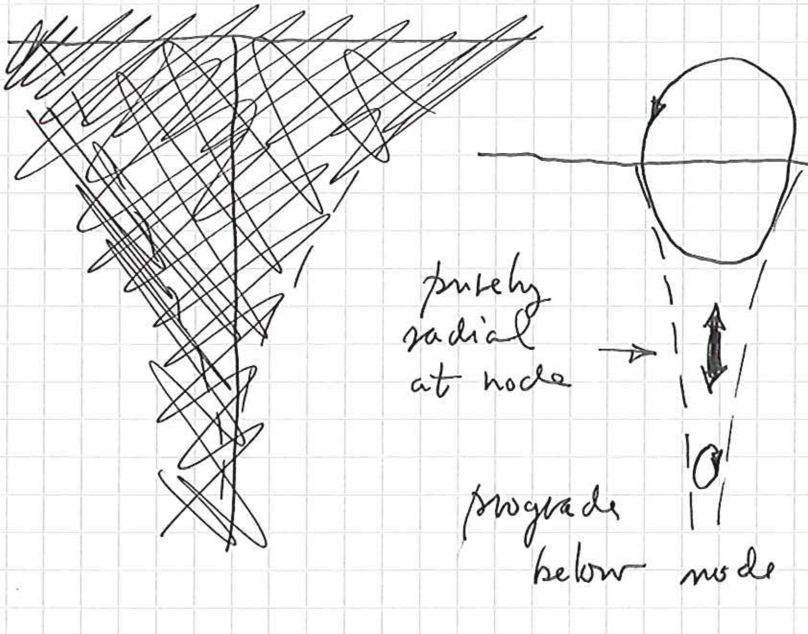


Fig. 10. – The mode $_0S_{19}$: displacement scalars U (——) and V (- - -), and perturbation potential Φ (— — —).

Fundamental mode $_0S_{19}$ ($T \sim 300s$)

At the surface $U(a)$ & $V(a)$ have opposite signs. The factor $-i\hat{\Delta}$ in $[\hat{r}U(r) - ika\hat{\Delta}V(r)]$ indicates that the $-\hat{\Delta}$ component lags the $\hat{r}$ component by 90°



well-known retrograde elliptical motion at the surface.