Small oscillations : DT Sect 7.1 , Goldstein 2nd ed., ch. 6

coordinates $q_1, \dots, q_N$
velocities $\dot{q}_1, \dots, \dot{q}_N$

kinetic energy quadratic in velocities

$$T = \tfrac{1}{2} T_{ij}(q_1, \dots, q_N)\, \dot{q}_i \dot{q}_j \quad , \quad T_{ij} = T_{ji}$$

also assume $T > 0$ if at least one $\dot{q}_i \neq 0$

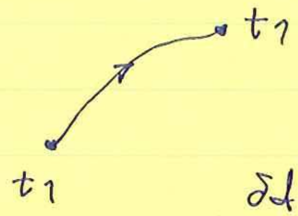
potential energy $V = V(q_1, \dots, q_N)$
conservative; see p. 8.

Lagrangian $L = T - V$

$L(q_1, \dots q_N ; \dot{q}_1, \dots \dot{q}_N)$

Hamilton's principle $L(\mathbf{q}, \dot{\mathbf{q}})$
vectors $\mathbf{q}, \dot{\mathbf{q}}$

action $$I = \int_{t_1}^{t_2} L \, dt$$



fixed endpoints $q_i(t_1)$, $q_i(t_2)$

$$\delta I = \delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})\, dt = 0$$

$$\delta I = \int_{t_1}^{t_2} \Big[\frac{\partial L}{\partial \mathbf{q}} \cdot \delta \mathbf{q} + \underbrace{\frac{\partial L}{\partial \dot{\mathbf{q}}}}_{u} \cdot \underbrace{\delta \dot{\mathbf{q}}}_{dv} \Big]\, dt$$

$$= \cancel{\left[\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \delta \mathbf{q} \right]_{t_1}^{t_2}} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) \right] \cdot \delta \mathbf{q} \, dt$$

zero, since
endpoints fixed

Euler–Lagrange : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0$

Go to page 8; conservation of energy here

Second-order non linear

Now suppose $\mathbf{q} = \mathbf{0}$ is an equil. configuration

$$\frac{\partial V}{\partial q_i} = 0 \quad \text{at} \quad q_i = 0$$

Small oscillations

$$T = \frac{1}{2} T_{ij}(\mathbf{q})\, \dot{q}_i \dot{q}_j$$

$$T_{ij}(\mathbf{q}) = T_{ij}(\mathbf{0}) + \cancel{\frac{\partial T_{ij}}{\partial q_k}(\mathbf{0})}\, q_k + \cdots$$

$$= T_{ij}$$

$$V(\mathbf{q}) = V(\mathbf{0}) + \underbrace{\cancel{\frac{\partial V}{\partial q_i}}}_{\text{zero}}(\mathbf{0})\, q_i + \frac{1}{2} \frac{\partial^2 V}{\partial q_i \partial q_j}(\mathbf{0})\, q_i q_j$$

Define matrices $T = (T_{ij})$, $V = (V_{ij})$

both real symmetric $\quad = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\mathbf{0}} \right)$

$$V = V_0 + \tfrac{1}{2}\, \underline{q}^T V \underline{q}$$

$$T = \tfrac{1}{2}\, \dot{\underline{q}}^T T \dot{\underline{q}}$$

$$L = T - V = \tfrac{1}{2}\left(\dot{\underline{q}}^T T \dot{\underline{q}} - \underline{q}^T V \underline{q} \right)$$

Euler–Lagrange $\qquad T\ddot{\underline{q}} + V\underline{q} = 0$

T, V real symmetric
T positive definite : $\dot{\underline{q}}^T T \dot{\underline{q}} > 0$
for all $\dot{\underline{q}} \neq 0$

Thus T^{-1} exists

comment on FT sign convention

Normal mode solutions $\quad \underline{q}(t) = \underline{q}\, e^{i\omega t}$

$V\underline{q} = \omega^2 T \underline{q}$, generalized eigenvalue problem

$$(T^{-1}V)\,\underline{q} = \omega^2 \underline{q}$$

$H = T^{-1}V \qquad H\underline{q} = \omega^2 \underline{q}$, ordinary

But H is not symmetric : $H^T = V T^{-1} \neq H$.

Nice theorems don't apply, e.g. ω^2 may be complex. But there's an easy fix

Introduce new (kinetic energy) inner product

$$\langle \underline{q}, \underline{q}' \rangle = \underline{q}^T T \underline{q} =$$

Then H is Hermitian w.r.t. $\langle , \rangle$

$$\langle \underline{q}, H\underline{q}' \rangle = \underline{q}^T T (T^{-1} V \underline{q}')$$

$$= \underline{q}^T V \underline{q}' = \underline{q}'^T V \underline{q}$$

$$= \underline{q}'^T T (T^{-1} V \underline{q}) = \langle \underline{q}', H\underline{q} \rangle$$

Eigenvalues <u>are</u> real

$$H\underline{q} = \omega^2 \underline{q} \quad , \quad H\underline{q}' = \omega'^2 \underline{q}'$$

$$\langle \underline{q}, H\underline{q}' \rangle = \langle \underline{q}, \omega'^2 \underline{q}' \rangle = \omega'^2 \langle \underline{q}, \underline{q}' \rangle$$

$$\langle \underline{q}', H\underline{q} \rangle = \langle \underline{q}', \omega^2 \underline{q} \rangle = \omega^2 \langle \underline{q}', \underline{q} \rangle$$
$$= \omega^2 \langle \underline{q}, \underline{q}' \rangle$$

$$(\omega^2 - \omega'^2) \langle \underline{q}, \underline{q}' \rangle = 0$$

$$\langle \underline{q}, \underline{q}' \rangle = 0 \text{ if } \omega^2 \neq \omega'^2 \quad \text{orthogonality}$$

May be repeated roots; can choose $\underline{q}$ orthogonal in degenerate eigenspace

normalization $\langle \underline{q}, \underline{q} \rangle = 1$ or $\underline{q}^T T \underline{q} = 1$

eigenvector matrix

$$Q = \begin{pmatrix} \vdots & & \vdots \\ \underline{q}_1 & & \underline{q}_N \\ \vdots & & \vdots \end{pmatrix}$$

$$Q^T T Q = I$$

$$\begin{pmatrix} \text{—} \\ \\ \end{pmatrix}\begin{pmatrix} & T & \\ \end{pmatrix}\begin{pmatrix} | & \\ & \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$VQ = TQ\Omega^2 ,$$

why on right?

$$\Omega^2 = \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_N^2 \end{pmatrix}$$

$$Q^T V Q = \Omega^2 , \qquad Q^T T Q = I$$

simultaneous diagonalization by congruent transformation Q

$$VQ = TQ\Omega^2$$
$$(T^{-1}V)Q = Q\Omega^2$$

similarity transform

$$Q^{-1}(T^{-1}V)Q = Q^{-1}HQ = \Omega^2$$

Note that $Q^{-1} \neq Q^T$, Q <u>not</u> an orthogonal transformation

Initial value problem:

find $\mathbf{q}(t)$ given $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$

normalized eigenvectors $\mathbf{e}_1, \dots \mathbf{e}_N$

normal modes

general solution a sum of $\mathbf{e}_i e^{\pm i\omega_i t}$

$$\underline{q}(t) = \sum_{n=1}^{N} \left[A_n \cos\omega_n t + \omega_n^{-1} B_n \sin\omega_n t \right] e_n$$

$$\underline{q}_0 = \underline{q}(0) = \sum_{n=1}^{N} A_n e_n$$

$$\dot{\underline{q}}_0 = \dot{\underline{q}}(0) = \sum_{n=1}^{N} B_n e_n$$

$$A_n = \langle e_n, \underline{q}_0 \rangle \quad , \quad B_n = \langle e_n, \dot{\underline{q}}_0 \rangle \omega_n^{-1}$$

$$\underline{q}(t) = \sum_{n=1}^{N} e_n \left[\langle e_n, \underline{q}_0 \rangle \cos\omega_n t + \omega_n^{-1} \langle e_n, \dot{\underline{q}}_0 \rangle \sin\omega_n t \right]$$

$$= \sum_{n=1}^{N} C_n e_n \cos(\omega_n t + \phi_n)$$

$$C_n \cos\phi_n = \langle e_n, \underline{q}_0 \rangle$$

$$C_n \sin\phi_n = -\omega_n^{-1} \langle e_n, \dot{\underline{q}}_0 \rangle$$

Shapes of normal mode oscillations are completely determined; amplitudes and phases are determined by the i.c.

Green ~~xxxxx~~ matrix

$$T\ddot{G} + VG = I\,\delta(t)$$

or $\quad T\ddot{G} + VG = 0 \qquad G(0) = 0, \ \dot{G}(0) = T^{-1}$

$$G(t) = Q\cos(\Omega t)A + Q\sin(\Omega t)B$$

$$QA = 0 \qquad Q\Omega B = T^{-1}$$

multiply on left by $Q^T T$

$$A = 0 \qquad B = \Omega^{-1} Q^T$$

$$G(t) = Q\Omega^{-1} \sin(\Omega t) Q^T H(t)$$

$\sin \Omega t$ is diagonal so $G = G^T$ source–receiver reciprocity

Response to arbitrary forcing: convolve with Green matrix

$$T\ddot{\underline{q}} + V\underline{q} = f(t)$$

$$\underline{q}(t) = \int_{-\infty}^{t} G(t-t') f(t')\, dt'$$

$$\dot{\underline{q}}(t) = \cancel{G(0) f(t)} + \int_{-\infty}^{t} \dot{G}(t-t') f(t')\, dt'$$

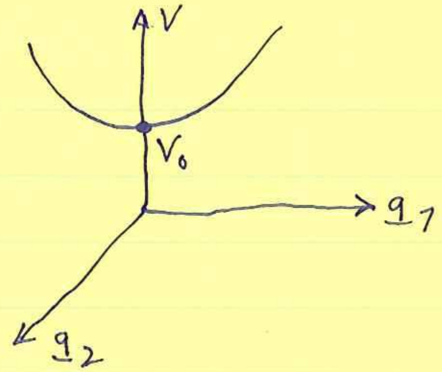
$$\ddot{\underline{q}}(t) = \dot{G}(0) f(t) + \int_{-\infty}^{t} \ddot{G}(t-t') f(t')\, dt'$$

$$T\ddot{\underline{q}} + V\underline{q} = TT^{-1} f(t) + \int_{-\infty}^{t} \cancel{[T\ddot{G} + VG]}(t-t') f(t')\, dt'$$

$$= f(t), \text{ check}$$

Linear stability analysis: ω^2 is real but may have either sign

If all $\omega_n^2 > 0$ then $V = V_0 + \frac{1}{2} \underline{q}^T V \underline{q}$ is a local <u>minimum</u>:

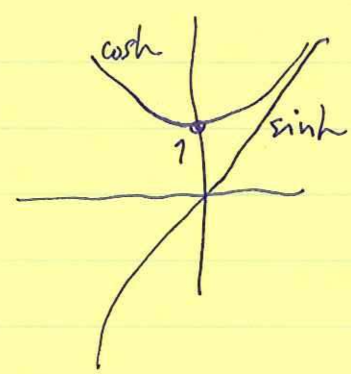


Suppose some eigenfrequency squared, $\omega_1^2 < 0 \Rightarrow \omega_1 = ip_1$

$\cos \omega_1 t = \cosh p_1 t$

$\omega_1^{-1} \sin \omega_1 t = p_1^{-1} \sinh p_1 t$

exponential instability



Plot of $V(\underline{q})$ a saddle

Conservation of energy: general Lagrangian $L(\underline{q}, \dot{\underline{q}}, t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) + \frac{\partial L}{\partial t} = 0$$

In our case $T = \frac{1}{2} q_i T_{ij}(\underline{q}) q_j + V(\underline{q})$

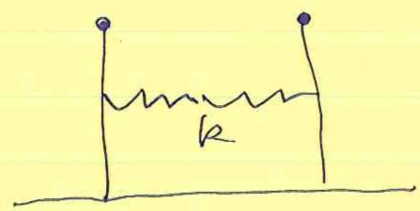
$$\frac{\partial L}{\partial t} = 0$$

also $\quad \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T$

$$\frac{d}{dt}(T+V) = 0$$ conservation of energy

Now do double pendulum example
pages 12-16 1982 notes

Inverted pendulum



$g \to -g$

$\omega_1 = \sqrt{-\frac{g}{l}}$ unstable

$\omega_2 = \sqrt{-\frac{g}{l} + \frac{2k}{m}}$ can be stable
if spring is
strong enough

Hard to avoid

made it to here
end of day 1 — 2 hours

Day 2 review

$$\underline{q}(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_N(t) \end{pmatrix} \qquad \dot{\underline{q}}(t) = \begin{pmatrix} \dot{q}_1(t) \\ \vdots \\ \dot{q}_N(t) \end{pmatrix}$$

$T\ddot{\underline{q}} + V\underline{q} = 0$, T, V symmetric, T pos. def

$$\frac{d}{dt}(T+V) = 0$$

$$\underset{ke}{T = \frac{1}{2}\dot{\underline{q}}^T T \dot{\underline{q}}} , \quad \underset{pe}{V = \frac{1}{2}\underline{q}^T V \underline{q}}$$

normal mode solutions: $\underline{q}(t) = \underline{q}\, e^{i\omega t}$
¿ why?

$V\underline{q} = \omega^2 T \underline{q}$, generalized eigenvalue problem

ke inner product $\langle \underline{q}, \underline{q}' \rangle = \underline{q}^T T \underline{q}'$

Then $\langle \underline{q}, H\underline{q}' \rangle = \langle H\underline{q}, \underline{q}' \rangle$, $H = T^{-1}V$

simultaneous diagonalization by a congruent transformation

$$Q = \begin{pmatrix} | & & | \\ \underline{q}_1 & & \underline{q}_N \\ | & & | \end{pmatrix}$$

$$Q^T T Q = I \quad , \quad Q^T V Q = \Omega^2$$

$$\Omega^2 = \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_N^2 \end{pmatrix}$$

orthonormality w.r.t. $\langle , \rangle$

$G(t) = \sum_{n=1}^{N} \omega_n^{-1} (q_n q_n^T) \sin \omega_n t$

Also Green tensor $G(t) = Q\, \Omega^{-1} \sin \Omega t\, Q\, H(t)$

$$Q^{-1} H Q = Q^{-1} (T^{-1} V) Q = \Omega^2$$

similarity transform

One question: how do we know Q is invertible?

source-receiver reciprocity: $G^T(t) = G(t)$

Alternative approach — closer to optimal numerical technique

$$T = R^2 \leftarrow \text{square root of } T \quad , \quad R^T = R$$

$$V\underline{q} = \omega^2 R^2 \underline{q} = \omega^2 R (R\underline{q})$$

$$R^{-1} V (R^{-1} R) \underline{q} = (R^{-1} V R^{-1})(R\underline{q}) = \omega^2 (R\underline{q})$$

Let $y = R\underline{q}$ $\qquad \underbrace{(R^{-1} V R^{-1})}_{\text{symmetric}} y = \omega^2 y$

now an ordinary eigenvalue problem

$$Y = \begin{pmatrix} | & & | \\ y_1 & \cdots & y_N \\ | & & | \end{pmatrix} \qquad \text{Then} \qquad Y = RQ$$

$$Y^T Y = I$$
$$\Rightarrow Y^{-1} = Y^T$$
orthogonal

$$Y^T (R^{-1} V R^{-1}) Y = \Omega^2$$

$$Y^T Y = Q^T R^T R Q = Q^T (R^2) Q$$
$$= Q^T T Q = I \quad \text{re orthonormality}$$

$$Q^T R^T R^{-1} V R^{-1} R Q$$
$$= Q^T V Q = \Omega^2$$

but also $Y = RQ = Q = \underbrace{R^{-1} Y}_{\text{both invertible}}$

$$\Rightarrow Q^{-1} = Y^{-1} R \quad \exists$$

$$Y^{-1} (R^{-1} V R^{-1}) Y$$
$$= Q^{-1} R^{-1} (R^{-1} V R^{-1}) R Q$$
$$= Q^{-1} (R^{-2} V) Q$$
$$= Q^{-1} (T^{-1} V) Q = Q^{-1} H Q = \Omega^2$$

Numerically — use a cholesky decomposition $T = LL^T$ instead.

Rayleigh's principle   D&T p. 113

$$H\underline{q} = \omega^2 \underline{q} \quad , \quad H = T^{-1} V$$

$$\omega^2 = \frac{\langle \underline{q}, H\underline{q} \rangle}{\langle \underline{q}, \underline{q} \rangle} = \frac{\underline{q}^T V \underline{q}}{\underline{q}^T T \underline{q}}$$ Rayleigh quotient

potential energy over kinetic energy

eigenvalue ω^2 is stationary for arbitrary $\delta\underline{q}$ iff $\underline{q}$ is an eigenvector with associated eigenvalue ω^2

Write schematically as $\omega^2 = \frac{\mathcal{V}}{\mathcal{T}}$

$$\delta\omega^2 = \delta\left(\frac{\mathcal{V}}{\mathcal{T}}\right) = \frac{\delta\mathcal{V}}{\mathcal{T}} - \frac{\mathcal{V}}{\mathcal{T}^2}\delta\mathcal{T}$$

$$= \frac{\delta\mathcal{V} - \omega^2 \delta\mathcal{T}}{\mathcal{T}} = \frac{1}{\mathcal{T}}\delta(\mathcal{V} - \omega^2\mathcal{T})$$

* later next page

In normal mode context convenient to define a freq domain action

$$\mathcal{A} = \frac{1}{2}(\omega^2\mathcal{T} - \mathcal{V}) = \frac{1}{2}\omega^2(\underline{q}^T T \underline{q}) - \frac{1}{2}\underline{q}^T V \underline{q}$$

$$\delta\omega^2 = -\frac{\delta\mathcal{A}}{2\langle \underline{q}, \underline{q}\rangle} = \; = -\frac{\delta\mathcal{A}}{2\underline{q}^T T \underline{q}}$$

ω^2 is stationary if $\mathcal{A}$ is and vice-versa

$$\delta(\mathcal{V} - \omega^2 \mathcal{T}) = \delta(\underline{q}^T V \underline{q} - \omega^2 \underline{q}^T T \underline{q})$$

$$= 2\,\delta \underline{q}^T (V\underline{q} - \omega^2 T \underline{q}) = 0 \quad \text{iff}$$

$$V\underline{q} - \omega^2 T \underline{q}$$

* from previous page here

Yet a third version: $\mathcal{V} = \underline{q}^T V \underline{q}$ is stationary subject to the normalization constraint $\mathcal{T} = \underline{q}^T T \underline{q} = 1$

stationarity of ω^2 physically appealing

We know 2 things about $\mathcal{L}$:

1. stationary
2. equal to zero $\mathcal{L} = \frac{1}{2}(\omega^2 \mathcal{T} - \mathcal{V}) = 0$ at stationary point

Now suppose that $\mathbf{T}$ and $\mathbf{V}$ depend on parameters $\mathbf{p}$, e.g. k, l, m for the pendula

$$\mathcal{L} = \mathcal{L}(\omega, \mathbf{q}, \mathbf{p}) = 0$$

Now suppose

$$\mathbf{p} \to \mathbf{p} + \delta \mathbf{p}, \quad \omega \to \omega + \delta\omega, \quad \mathbf{q} \to \mathbf{q} + \delta \mathbf{q}$$

$$\lambda(\omega, \mathbf{q}, \mathbf{p}) = \lambda(\omega+\delta\omega, \mathbf{q}+\delta\mathbf{q}, \mathbf{p}+\delta\mathbf{p}) = 0$$

Take the <u>total</u> variation w.r.t. all arguments

$$\delta_{total}\, \tfrac{1}{2}\left(\omega^2 \mathbf{q}^T \mathbf{T} \mathbf{q} - \mathbf{q}^T \mathbf{V} \mathbf{q}\right)$$

$$= \tfrac{1}{2}\delta\omega^2\, \mathbf{q}^T \mathbf{T} \mathbf{q} + \cancel{\delta\mathbf{q}^T\left(\omega^2 \mathbf{T}\mathbf{q} - \mathbf{V}\mathbf{q}\right)}$$

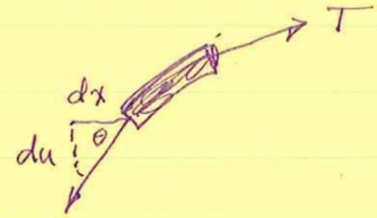
$$+ \tfrac{1}{2}\left(\omega^2 \mathbf{q}^T \delta\mathbf{T}\, \mathbf{q} - \mathbf{q}^T \delta\mathbf{V}\, \mathbf{q}\right) = 0$$

$$\delta\omega^2 = \frac{\mathbf{q}^T \delta\mathbf{V}\, \mathbf{q} - \omega^2 \mathbf{q}^T \delta\mathbf{T}\, \mathbf{q}}{\mathbf{q}^T \mathbf{T} \mathbf{q}}$$

Double pendulum example – page 28 of 1982 notes.

<u>Violin string:</u> tension T, density $\rho(x)$

$u(x,t)$ only in y direction



$$\tan\theta \approx \sin\theta = \frac{\partial u}{\partial x}$$

restoring force $\quad T\,[\,\tan\theta(x+dx) - \tan\theta(x)\,]$

$$\approx T\left[\frac{\partial u}{\partial x}(x+dx) - \frac{\partial u}{\partial x}(x)\right]$$

Newton's law: $F = ma$

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} = T\left[\frac{\frac{\partial u}{\partial x}(x+dx) - \frac{\partial u}{\partial x}(x)}{dx}\right]$$

$$\rho(x)\,\partial_t^2 u = T\,\partial_x^2 u$$

fixed ends $u(0,t) = u(L,t) = 0$

Multiply by velocity $\partial_t u$ and $\int_0^L dx$

$$\text{lhs} = \int_0^L \rho\,\partial_t^2 u\,\partial_t u\,dx = \frac{d}{dt}\,\frac{1}{2}\int_0^L \rho(\partial_t u)^2\,dx$$

rhs by parts

$$T\int_0^L \partial_x^2 u\ \partial_t u\ dx = T\left[\cancel{\partial_x u\ \partial_t u}\right]_0^L$$
zero

$$-T\int_0^L \partial_x u\ \frac{\partial}{\partial t}(\partial_x u)\ dx$$

$$= -\frac{d}{dt}\frac{1}{2}\int_0^L T(\partial_x u)^2\,dx$$

$$\frac{d}{dt}\int_0^L \left[\frac{1}{2}\rho(\partial_t u)^2 + \frac{1}{2}T(\partial_x u)^2\right]dx = 0$$

conservation of energy

$\frac{1}{2}\rho(\partial_t u)^2$: ke density
$\frac{1}{2}T(\partial_x u)^2$: pe density

Hamilton's principle :

$$I = \int_{t_1}^{t_2}\int_0^L L(u, \partial_t u, \partial_x u)\ dx\, dt$$

$$\delta I = 0 \qquad L = \frac{1}{2}\rho(\partial_t u)^2 - \frac{1}{2}T(\partial_x u)^2$$

$$I = \int_{t_1}^{t_2}\int_0^L \left[\frac{1}{2}\rho(\partial_t u)^2 - \frac{1}{2}T(\partial_x u)^2\right] dx\ dt$$

$$\delta I = \delta\int_{t_1}^{t_2} dt\ \int_0^L L(u, \partial_t u, \partial_x u)\ dx$$

$$= \int_{t_1}^{t_2} dt\ \int_0^L \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial(\partial_t u)}\partial_t\delta u + \frac{\partial L}{\partial(\partial_x u)}\partial_x\delta u\right)dx$$

$$= \int_0^L \left[\frac{\partial \mathcal{L}}{\partial(\partial_t u)} \delta u \right]_{t_1}^{t_2} dx$$

$\to 0$ since $u(x,t_1)$ and $u(x,t_2)$ fixed

$$+ \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial(\partial_x u)} \delta u \right]_0^L dt$$

$\to 0$ see below

$$+ \int_{t_1}^{t_2} dt \int_0^L \delta u \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t u)} \right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x u)} \right) \right] dx$$

~~fixed ends~~

natural bc is $\frac{\partial \mathcal{L}}{\partial(\partial_x u)} = T(\partial_x u) = 0$ free ends

instead we must impose an admissibility constraint upon δu : namely $\delta u(0,t) = \delta u(L,t) = 0$

Then I stationary for arbitrary <u>admissible</u> δu iff

$$\cancel{\frac{\partial \mathcal{L}}{\partial u}} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t u)} \right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x u)} \right) = 0$$

zero in our case

$$\rho\, \partial_t^2 u - T \partial_x^2 u = 0$$

Normal mode solutions: $u(x,t) = u(x)\, e^{i\omega t}$

eigenfcn: mode shape

eigenfreq

comment on FT convention $\neq$ Aki & Richards

$$-T\frac{d^2u}{dx^2} = \rho\omega^2 u \qquad \text{with } u(0)=u(L)=0$$

eigenvalue problem, $\rho>0$ analogous to π pos-def.

the inner product

$$\langle u, u'\rangle = \int_0^L \rho(x)\,u(x)\,u'(x)\,dx$$

Rewrite as

$$-\rho^{-1}T\frac{d^2u}{dx^2} = \omega^2 u\ , \quad H = -\rho^{-1}T\frac{d^2}{dx^2}$$

Operator $H = -\rho^{-1}T\frac{d^2}{dx^2}$ Hermitian w.r.t. $\langle\, ,\, \rangle$

Proof easy: 2 integrations by parts

$$\langle u, \overbrace{-\rho^{-1}T\frac{d^2}{dx^2}}^{H} u'\rangle = -T\int_0^L u\,\frac{d^2u'}{dx^2}\,dx$$

$$= \underbrace{-T\left[u\frac{du'}{dx}\right]_0^L}_{\text{zero by bc.}} + T\int_0^L \frac{du}{dx}\frac{du'}{dx}\,dx$$

$$= \underbrace{T\left[\frac{du}{dx}u'\right]_0^L}_{\text{ditto}} - T\int_0^L \frac{d^2u}{dx^2}\,u'\,dx$$

$$= \langle \overbrace{-\rho^{-1}T\frac{d^2}{dx^2}}^{H} u, u'\rangle \qquad \text{qed}$$

intimately involves bc — typical of field theories

Now easy to show that ω^2 real and

$$(\omega^2-\omega'^2)\langle u,u'\rangle = (\omega^2-\omega'^2)\int_0^L \rho u u' dx = 0$$

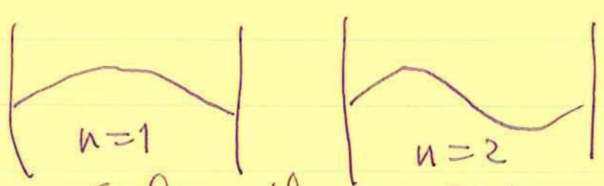
orthogonality   end here class #2

intimate association: Hermitian, real eigenvalues, energy conservation

Example: $\rho(x)=\rho$, homogeneous

$$-(T/\rho)\frac{d^2u}{dx^2} = \omega^2 u, \qquad u(0)=u(L)=0$$

$$\left.\begin{aligned} \omega_n &= \frac{n\pi}{L}\sqrt{T/\rho} \\ u_n(x) &= \sqrt{\frac{2}{\rho L}}\,\sin\left(\frac{n\pi x}{L}\right) \end{aligned}\right\} n = 1, 2, \ldots$$



fundamental   first overtone at twice frequency

orthogonality $\quad \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,\sin\frac{m\pi x}{L}\,dx = \delta_{nm}$

$$\langle u_n, u_m\rangle = \delta_{nm}$$

Class #3

Hunb's question — when is it useful to seek normal mode solutions?

$$\rho(x)\, \partial_t^2 u = T\, \partial_x^2 u$$

b.c. $u(0,t) = u(L,t) = 0$

Instead seek a separable solution

$$u(x,t) = f(x)\, g(t)$$

$$\rho f \frac{d^2 g}{dt^2} = T g \frac{d^2 f}{dx^2}$$

$$\underbrace{\frac{1}{Tg} \frac{d^2 g}{dt^2}}_{\text{fcn of } t \text{ only}} = \underbrace{\frac{1}{\rho f} \frac{d^2 f}{dx^2}}_{\text{fcn of } x \text{ only}}$$

$\Rightarrow$ must both be = to a constant, which we call $-\omega^2/T$

$$\underbrace{\frac{1}{g}\frac{d^2 g}{dt^2} = -\omega^2}_{\Rightarrow\ g \sim e^{i\omega t}} = \frac{T}{\rho}\frac{1}{f}\frac{d^2 f}{dx^2}$$

$$T \frac{d^2 f}{dx^2} = -\rho\omega^2 f$$

But what if $T = T(t)$: tuning your violin while you play it.

Then $\frac{1}{T(t)} \frac{1}{g} \frac{d^2 g}{dt^2} = \text{const} = \frac{1}{\rho(x)} \frac{1}{f} \frac{d^2 f}{dx^2}$

In general it is permissible to to seek normal mode solutions whenever none of the parameters describing the system or model depend on ~~the~~ time

In the discrete case

$$\mathbb{T} \ddot{\underline{q}} + \mathbb{V} \underline{q} = 0$$

ke: $T = \frac{1}{2} \dot{\underline{q}}^T \mathbb{T} \dot{\underline{q}}$

pe: $V = V_0 + \frac{1}{2} \underline{q}^T \mathbb{V} \underline{q}$

$\mathbb{T}$ and $\mathbb{V}$ are time-independent

This the hallmark of a <u>conservative</u> <u>system</u>.

Then do <u>homogeneous</u> string

4th lecture: Thurs Feb 15

tensors as multilinear functionals
taken from mimeographed notes
plus Appendix A of D & T.

5th class : 20 Feb

linear functional $f(\underline{v}) \to$ scalar
slot

isomorphic to vectors
$f(\underline{v}) = \underline{f} \cdot \underline{v}$ for all $\underline{v}$

multilinear functional $T(\underline{u}, \underline{v})$

tensor product : $TS(\underline{u}, \underline{v}, \underline{x}, \underline{y})$
$= T(\underline{u}, \underline{v}) S(\underline{x}, \underline{y})$ order $p+\underline{q}$

write without $\otimes$ $T \otimes S$

trace $\operatorname{tr} T = T(\hat{x}_i, \hat{x}_i) = T(\hat{x}_i', \hat{x}_i')$

transpose $T^T(\underline{u}, \underline{v}) = T(\underline{v}, \underline{u})$

components $T_{ij} = T(\hat{x}_i, \hat{x}_j)$
just like $f_i = \underline{f} \cdot \hat{x}_i = f(\hat{x}_i)$

$\hat{x}_{i_1} \cdots \hat{x}_{i_q}$ a basis for space of tensors of order q - dimension 3^q

e.g. $T = T_{ij} \hat{x}_i \hat{x}_j$

$(TS)_{ijkl} = T_{ij} S_{kl}$

$\operatorname{tr} T = T_{ii}$

identity tensor $\underline{\underline{I}}(\underline{u}, \underline{v}) = \underline{u}\cdot\underline{v})$

$I_{ij} = \delta_{ij}$, isotropic tensor

alternating tensor $\Lambda(\underline{u}, \underline{v}, \underline{w}) = \underline{u}\cdot(\underline{v}\times\underline{w})$

$\Lambda_{ijk} = \pm\, \varepsilon_{ijk}$ rt handed / left handed

Most general isotropic tensor $a\underline{\underline{I}}$, $a\,\delta_{ij}$

Change of basis : $T'_{ij} = (\hat{x}'_i\cdot\hat{x}_j)(\hat{x}'_j\cdot\hat{x}_k)\, T_{kl}$

Then : tensors of order $q=2$ as linear operators

End with physical examples of tensors. Common in linear constitutive relations

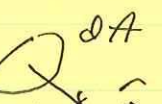
Ohm's law : $\underline{J} = \underline{\underline{\sigma}}\cdot\underline{E}$

thermal conductivity : $\underline{H} = -\underline{\underline{K}}\cdot\nabla\theta$

dielectric tensor : $\underline{D} = \underline{\underline{\varepsilon}}\cdot\underline{E}$

inertia tensor : $\underline{L} = \underline{\underline{I}}\cdot\underline{\omega}$

stress tensor :



$\underline{f} = (\hat{n}\, dA)\cdot\underline{\underline{T}}$

or since $\underline{\underline{T}} = \underline{\underline{T}}^T$

$\underline{f} = \underline{\underline{T}}\cdot(\hat{n}\, dA)$

Rayleigh's principle:

$$H = -\rho^{-1}(x)\, T \frac{d^2}{dx^2}$$

$$Hu = \omega^2 u$$

$$\omega^2 = \frac{\langle u, Hu \rangle}{\langle u, u \rangle} = \frac{T \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx}{\int_0^L \rho u^2 \, dx} = \frac{V}{\mathcal{T}}$$

$$\delta\omega^2 = \frac{1}{\mathcal{T}}\, \delta(V - \omega^2 \mathcal{T})$$

normal mode action:

$$I = \frac{1}{2}(\omega^2 \mathcal{T} - V) = \frac{1}{2}\int_0^L \left[\rho\omega^2 u^2 - T\left(\frac{\partial u}{\partial x}\right)^2\right] dx$$

$\delta\omega^2 = 0$ iff $\delta I = 0$ as before

$$\delta I = \int_0^L \delta u \left(\rho\omega^2 u + T\frac{d^2 u}{dx^2}\right) dx$$
$$- \left[\delta u \left(T \frac{du}{dx}\right)\right]_-^+ = 0$$

again need admissibility constraint $\delta u = 0$ on ends

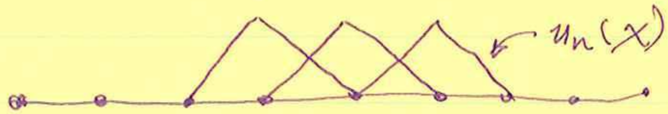
$\delta I = 0$ iff $-\rho^{-1} T \frac{d^2 u}{dx} = \omega^2 u$, $u(0) = u(L) = 0$

Application: Rayleigh-Ritz method — heterogeneous string, $\rho(x)$

Basis functions $\cancel{\phi_1(x), \ldots, \phi_N}$
$u_1(x), \ldots, u_N(x)$

e.g. global $u_n(x) = \sin \frac{n\pi x}{L}$

local – piecewise linear splines



$$u(x) = \sum_{n=1}^{N} q_n u_n(x), \quad q_n \text{ expansion coeffs.}$$

$$I = \frac{1}{2}(\omega^2 \mathcal{T} - \mathcal{V}) = \frac{1}{2}\int_0^L \left(\rho\omega^2 u^2 - T\left(\frac{du}{dx}\right)^2\right) dx$$

$$= \frac{1}{2}\omega^2 \mathbf{q}^T \mathbf{T} \mathbf{q} - \frac{1}{2}\mathbf{q}^T \mathbf{V} \mathbf{q}$$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

$$V_{ij} = \mathcal{V}(u_i, u_j) = T\int_0^L \frac{du_i}{dx}\frac{du_j}{dx}\, dx$$

$$T_{ij} = \mathcal{T}(u_i, u_j) = \int_0^L \rho(x)\, u_i(x)\, u_j(x)\, dx$$

$\mathbf{T} = \mathbf{T}^T$, $\mathbf{V} = \mathbf{V}^T$ and $\mathbf{T}$ pos. def.

Reduces to previously considered problem

e.g. with a global basis

$u_n(x) = \sin \frac{n\pi x}{L}$, satisfy bc

~~[illegible]~~ $V_{mn} = \frac{1}{2} n^2 \pi^2 (T/L)\, \delta_{mn}$, diagonal

$T_{mn} = \int_0^L \rho(x) \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx$

Easiest in this case (since T = const) to solve

$$V^{-1} T \underline{q} = \omega^{-2} \underline{q}$$

Second application: first-order perturbation

$$\rho \to \rho + \delta\rho \quad , \omega \to \omega + \delta\omega , \quad u \to u + \delta u$$

Method same as before

$$I(\omega, u, \rho) = I(\omega + \delta\omega, u + \delta u, \rho + \delta\rho) = 0$$

$$I = \frac{1}{2} \int_0^L \left[\rho \omega^2 u^2 - T \left(\frac{du}{dx}\right)^2 \right] dx = 0$$

$$\delta_{total} I = \frac{1}{2}\, \delta\omega^2 \int_0^L \rho u^2\, dx$$

$$+ \frac{1}{2} \int_0^L \delta\rho\, \omega^2 u^2\, dx$$

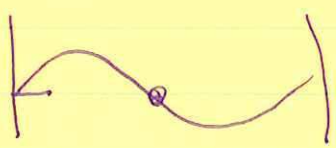
$$+ \underbrace{\int_0^L \delta u \left(\rho\omega^2 u + T \frac{d^2 u}{dx^2}\right) dx - \left[\delta u \left(T \frac{du}{dx}\right)\right]_-^+}_{\text{zero by Rayleigh}} = 0$$

$$\delta\omega^2/\omega^2 \cong \frac{2\omega}{\delta\omega} = -\frac{\int_0^L \delta\rho\, u^2\, dx}{\int_0^L \rho u^2\, dx}$$

e.g. a bead of mass m at $x = L/2$

$$\delta\rho = m\,\delta(x - L/2)$$

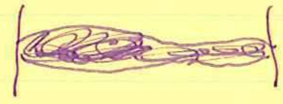
$$\frac{\delta\omega_n}{\omega_n} = -\left(\frac{m}{\rho L}\right)\sin^2\frac{n\pi}{2} = \begin{cases} -m/\rho L & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$



even:
at node

odd:
graver oscillation

Example 2: <u>smooth</u> continuous perturbation



0 L

$|\delta\rho| \ll \rho$

$$\frac{\delta\omega_n}{\omega_n} = -\frac{1}{\rho L}\int_0^L \delta\rho(x)\sin^2\frac{n\pi x}{L}\,dx$$

$$= -\frac{1}{2\rho L}\int_0^L \delta\rho(x)\left(1 - \cancel{\cos\frac{2n\pi x}{L}}\right)dx$$

zero for $n \gg 1$
high freq. mode

$$\omega_n \approx \frac{n\pi\sqrt{T/\rho}}{L}\left[1 - \frac{1}{2L}\int_0^L (\delta\rho/\rho)\,dx\right]$$

$c = \sqrt{T/\rho}$ is wavespeed

$$\omega_n = \frac{2n\pi}{T}, \quad T = \frac{2L}{c} \qquad \omega_n = \frac{n\pi c}{L}$$

generalized to

$$\omega_n \approx \frac{2n\pi}{T}, \quad T = 2\int_0^L \frac{dx}{c(x)}$$

in this case $\quad c = \sqrt{\frac{T}{\rho+\delta\rho}} \approx c\left[1 - \frac{1}{2}\frac{\delta\rho}{\rho}\right]$

Green function $g(x, x'; t)$ satisfies

~~$\rho \partial_t^2 g + T \partial_x^2 g =$~~

$$\rho \partial_t^2 g = T \partial_x^2 g + \delta(x-x')\delta(t)$$

equivalent initial value problem

$$\rho \partial_t^2 g = T \partial_x^2 g$$

$$\rho[\partial_t g]_-^+ = 1, \quad [g]_-^+ = 0$$

gen'l soln $g(x, x'; t) = \sum_n (a_n \cos\omega_n t + b_n \sin\omega_n t) u_n(x)$

$$\sum_n a_n u_n = 0, \quad \sum_n \omega_n b_n u_n = \rho^{-1}\delta(x-x')$$

orthogonality: $\quad a_n = 0, \quad b_n = \omega_n^{-1} u_n(x')$

amp. factor

shape of oscillation

$$g(x,x';t) = \sum_n \omega_n^{-1} \sin\omega_n t \; u_n(x')\, u_n(x)$$

heterogeneous string

oscillating in time

source-receiver reciprocity : $g(x,x';t) = g(x',x;t)$

Fourier domain:

comment on sign: opposite from A&R

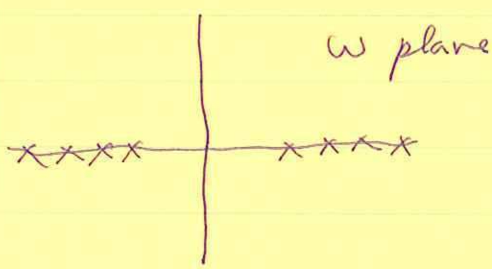
$$g(\omega) = \int_0^\infty g(t)\, e^{-i\omega t}\, dt$$

comment

$$g(x,x';\omega) = \sum_n \frac{u_n(x')\, u_n(x)}{\omega_n^2 - \omega^2}$$

Instructive to verify by working backward

$$g(x,x';t) = \sum_n u_n(x')\, u_n(x)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}\, d\omega}{(\omega_n - \omega)(\omega_n + \omega)}$$



poles at $\omega = \pm\omega_n$ on real axis

Two ways to eliminate ambiguity

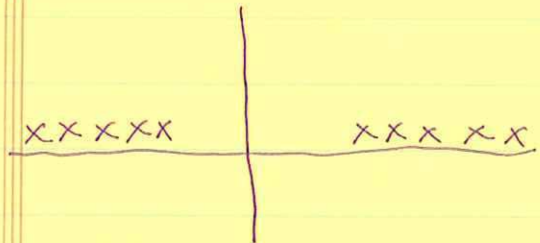
1. real systems <u>always</u> have dissipation

$\omega_n \to \omega_n + i\varepsilon_n , \quad \varepsilon_n > 0$

why + ?

since $e^{i\omega_n t} \to e^{i\omega_n t - \varepsilon_n t}$

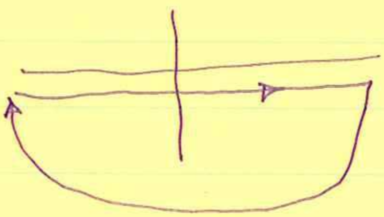
2. causality : response $\underline{\text{must}}$ be zero before $t=0$.



$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}\,d\omega}{(\omega-\omega_n)(\omega+\omega_n)}$$

$t<0$: close in lower halfplane $\text{Im}\,\omega<0$

$\Rightarrow e^{i\omega t}\to 0$ on arc



$$g(x,x';t)=0,\quad t\le 0$$

$t\geq 0$: close in upper halfplane, pick up poles

$$-\frac{1}{2\pi}(2\pi i)\,\Sigma\,\text{residues} = \frac{\sin\omega_n t}{\omega_n}$$

Finally, conversion to travelling waves (homog. string)

$$\omega_n = \frac{n\pi c}{L}\ ;\quad c=\sqrt{T/\rho}$$

$$u_n(x) = \sqrt{\frac{2}{\rho L}}\,\sin\frac{n\pi x}{L}$$

$$g(t) = \frac{2}{\rho c}\sum_{n=1}^{\infty}\frac{1}{n\pi}\underbrace{\sin(n\pi x'/L)}_{\text{amp. factor}}\ \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{shape}}\ \underbrace{\sin\left(\frac{n\pi c t}{L}\right)}_{\text{oscillatory in time}}$$

sum over modes

depends on source location and strength ; in this case 1 in FT domain

$$g(\omega) = \frac{2}{\rho L} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x'}{L} \sin \frac{n\pi x}{L}}{\frac{n^2\pi^2 c^2}{L^2} - \omega^2}$$

$$= \frac{L}{\rho \pi^2 c^2} \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{L}|x-x'| - \cos \frac{n\pi}{L}(x+x')}{n^2 - \omega^2 L^2/\pi^2 c^2}$$

use the Fourier series identity

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^2 - \alpha^2} = \underbrace{\frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos \alpha(\pi - z)}{\alpha \sin \alpha\pi}}_{\text{this is the Fourier cosine series of this function which satisfies } f(2\pi - z) = f(z)} , \quad 0 \le z \le 2\pi$$

$$g(\omega) = \frac{1}{2\rho c} \left\{ \frac{\cos \frac{\omega}{c}[L - (x+x')] - \cos \frac{\omega}{c}[L - |x-x'|]}{\omega \sin \frac{\omega L}{c}} \right\}$$

numerator $\cos \chi = \frac{1}{2}(e^{i\chi} + e^{-i\chi})$

denominator $\frac{1}{\sin \frac{\omega L}{c}} = 2i \left[e^{i\omega L/c} - e^{-i\omega L/c} \right]^{-1}$

$$= 2i\, e^{-i\omega L/c} \left[1 - e^{-2i\omega L/c} \right]^{-1}$$ geom. series

$$= 2i\, e^{-i\omega L/c} \sum_{m=0}^{\infty} e^{-2im\omega L/c}$$

$$d_1 = |x - x'| \qquad N_1 = 0$$
$$d_2 = x + x' \qquad N_2 = 1$$
$$d_3 = 2L - (x + x') \qquad N_2 = 1$$
$$d_4 = 2L - |x - x'| \qquad N_4 = 2$$
$$d_{j+4} = 2L + d_j \qquad N_{j+4} = N_j + 2$$

sketch picture of $d_1 - d_5$

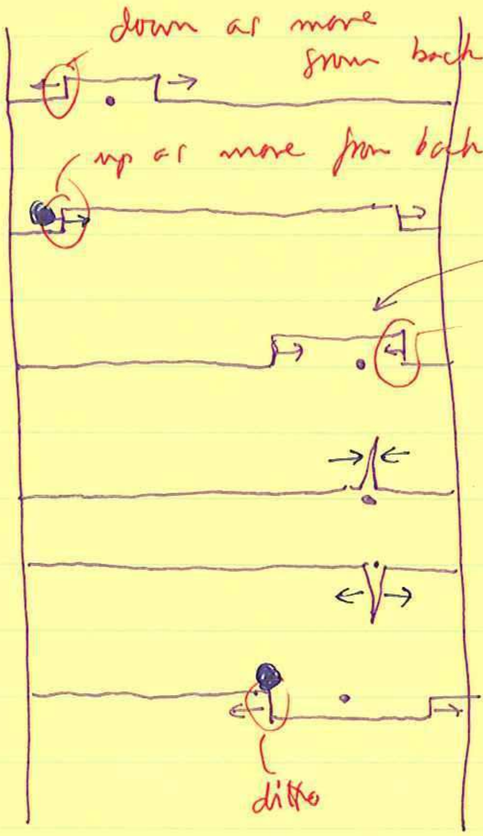
$$g(\omega) = \frac{1}{2\rho c} \sum_{j=1}^{\infty} (i\omega)^{-1} e^{-i\omega d_j / c - i N_j \pi}$$

$(-1)^{N_j}$ ← phase change upon reflection of ends

$$g(t) = \frac{1}{2\rho c} \sum_{j=1}^{\infty} e^{-i N_j \pi} H(t - d_j / c)$$

Heaviside function

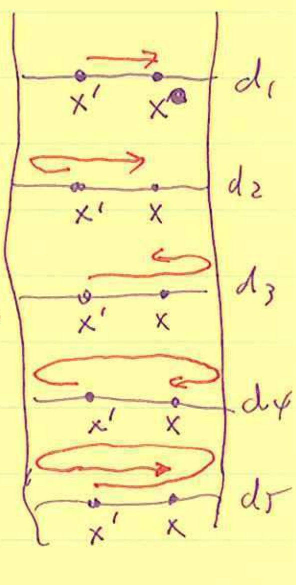
sum of propagating step pulses



antipode

up as move from back to front

both pull down



source on left

Local energy conservation law

$\partial_t s \cdot [\rho \partial_t^2 s = \nabla \cdot T]$

$$\partial_t \left(\tfrac{1}{2} \rho s \cdot s\right) - \underbrace{\partial_t s_j \cdot \partial_i \left(c_{ijkl}\, \partial_k s_l\right)}_{*} = 0$$

$$* = -\partial_i \left(c_{ijkl}\, \partial_k s_l\, \partial_t s_j\right) + c_{ijkl}\, \partial_k s_l\, \partial_t (\partial_i s_j)$$

$$= -\partial_i \left(T_{ij}\, \partial_t s_j\right) + \partial_t \left(\tfrac{1}{2} c_{ijkl}\, \partial_i s_j\, \partial_k s_l\right)$$

Get $\quad \partial_t E + \nabla \cdot K = 0$

$$E = \tfrac{1}{2} \rho (\partial_t s)^2 + \tfrac{1}{2} \varepsilon : c : \varepsilon$$

$$K = -\partial_t s \cdot T$$

Generic conservation law for ϕ-stuff

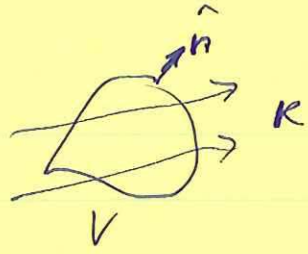
$$\partial_t \phi + \nabla \cdot K = k$$

ϕ, k order q

K order $q+1$

integrate over fixed volume V:

$$\underbrace{\frac{d}{dt} \int_V \phi \, dV}_{\text{rate of change of } \phi\text{-stuff}} = \underbrace{-\int_{\partial V} \hat{n} \cdot K \, dA}_{\text{flux}} + \underbrace{\int_V k \, dV}_{\text{creation of } \phi\text{-stuff}}$$



ϕ: density of ϕ-stuff
K: flux of ϕ-stuff (per unit area per sec)
k: creation rate of ϕ-stuff.

Momentum equation: $\partial_t(\rho s) + \nabla\cdot T = 0$

$\phi = \rho s$ momentum density (per unit vol.)

T = momentum flux

T_{ij} = flux of j component of momentum in i direction

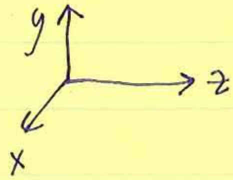
Energy: ϕ-stuff = energy

$$\phi = E = \underbrace{\tfrac{1}{2}\rho(\partial_t s)^2}_{\text{ke density}} + \underbrace{\tfrac{1}{2}\,\varepsilon:\Gamma:\varepsilon}_{\text{pe density}}$$

Energy flux in a linear elastic medium

$K = -\partial_t s\cdot T$ (minus velocity dot stress)

Example: plane P wave



$s_x = s_y = 0 \qquad s_z = A\sin(kz-\omega t)$

$$T_{ij} = (\kappa - \tfrac{2}{3}\mu)\,\nabla\cdot s\,\delta_{ij} + 2\mu(\partial_i s_j + \partial_j s_i)$$

$T_{xx} = T_{yy} = (\kappa - \frac{2}{3}\mu)\partial_z s_z$, $T_{zz} = (\kappa + \frac{4}{3}\mu)\partial_z s_z$, all rest zero

$$K_z = -\omega A[-\cos(kz-\omega t)]\,k(\kappa+\tfrac{4}{3}\mu)A\cos(kz-\omega t)$$
$$= \omega k(\kappa+\tfrac{4}{3}\mu)A^2\cos^2(kz-\omega t) = \rho\omega^2\alpha A^2\cos^2(kz-\omega t)$$
$$\langle K_z\rangle = \tfrac{1}{2}\rho\omega^2\alpha A^2$$