$\boxed{\text{First lecture}}$

$$H\underline{q} = \lambda \underline{q} \qquad H\underline{q}' = \lambda' \underline{q}'$$

$$\langle \underline{q}, H\underline{q}' \rangle = \langle \underline{q}, \lambda' \underline{q}' \rangle$$

$$= \lambda' \langle \underline{q}, \underline{q}' \rangle$$

$$\langle H\underline{q}, \underline{q}' \rangle = \lambda \langle \underline{q}, \underline{q}' \rangle$$

Thus $(\lambda - \lambda') \langle \underline{q}, \underline{q}' \rangle$

$N \times N$ has N <u>real</u> ω^2
may be repeated roots

$$Q = \begin{pmatrix} \vdots & & | \\ \underline{q}_1 & \cdots & \underline{q}_N \\ | & & | \end{pmatrix}$$

$$\boxed{Q^T T Q = I}$$

$$\begin{pmatrix} \text{---} \end{pmatrix} \begin{pmatrix} T \end{pmatrix} \begin{pmatrix} | \end{pmatrix} = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$V Q = T Q \Omega^2$$

$$= \begin{pmatrix} T \end{pmatrix} \begin{pmatrix} \underline{q} \end{pmatrix} \begin{pmatrix} \ddots & & \\ & \omega^2 & \\ & & \ddots \end{pmatrix}$$

$$\boxed{Q^T V Q = \Omega^2}$$

$$VQ = TQ\Omega^2$$

$$T^{-1}VQ = Q\Omega^2$$

$$Q^{-1}(T^{-1}V)Q = \Omega^2$$

$Q^{-1} \neq Q^T$ not orthogonal

Pendulum

l  $\omega^2 = l/g \qquad \omega = \pm\sqrt{l/g}$

Inverted pendulum  $\omega^2 = -l/g$

$$\omega = \pm i\sqrt{l/g}$$

$$\underline{q}(t) = \sum_n (a_n \cos\omega_n t + b_n \sin\omega_n t)\,\underline{q}_n$$

$$\underline{q}(0) = \sum_n a_n \underline{q}_n$$

$$\dot{\underline{q}}(0) = \sum_n \omega_n b_n \underline{q}_n$$

$$a_n = \langle \underline{q}_n, \underline{q}(0)\rangle$$

$$b_n = \omega_n^{-1}\langle \underline{q}_n, \dot{\underline{q}}(0)\rangle$$

$$\underline{q}(t) = \sum_n \underline{q}_n \left[\langle \underline{q}_n , \underline{q}(0) \rangle \cos\omega_n t \right.$$

$$\left. + \ \omega_n^{-1} \langle \underline{q}_n , \dot{\underline{q}}(0) \rangle \sin\omega_n t \right]$$

$$L = T - V = \frac{1}{2} \left(\dot{\underline{q}}^T T \dot{\underline{q}} - \underline{q}^T V \underline{q} \right)$$

$$\cancel{= \frac{1}{2} \left(\dot{q}_i T_{ij} \dot{q}_j - q_i V_{ij} q_j \right)}$$

$$= \frac{1}{2} \left(\dot{q}_j T_{jk} \dot{q}_k - q_j V_{jk} q_k \right)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

$$T_{jk} \ddot{q}_k - V_{jk} q_k = 0$$

$$T \dot{\underline{q}} - V \underline{q} = 0$$

$$V \underline{q} = \omega^2 T \underline{q}$$

$$\det \left(V - \omega^2 T \right) = 0$$

$$\rho\, \partial_t^2 u = T \partial_x^2 u + \delta(x - x_s)\, \delta(t)$$

$\boxed{\text{Second lecture}}$

$$\int_{-\varepsilon}^{\varepsilon} dt$$

$$\rho_s \left[\partial_t u\right]_-^+ = \delta(x - x_s)$$

$$\rho\, \partial_t^2 u = T \partial_x^2 u$$

$\rho^{-1}(x)$

$$u(0) = 0$$

$$\partial_t u(0) = \rho_s^{-1}\, \delta(x - X_s)$$

$$u(x,t) = \sum_n \left(a_n \cos\omega_n t + b_n \sin\omega_n t\right) u_n(x)$$

$$\sum_n a_n u_n = 0$$

$\rho^{-1}(x)$

$$\sum_n \omega_n b_n u_n = \rho_s^{-1}\, \delta(x - x_s)$$

$$a_n = 0$$

$$\omega_n b_n = u_n(x_s)$$

call this $g(x|x_s, t)$

$$\boxed{u(x,t) = \sum_n \omega_n^{-1} \sin\omega_n t\; u_n(x_s)\, u_n(x)}$$

Then $$\boxed{g(x|x_s, t) = g(x_s|x, t)}$$

reciprocity

FT $$u(x,\omega) = \int_0^\infty u(x,t)\, e^{-i\omega t}\, dt$$

comment on sign

$$\boxed{g(x|x_s,\omega) = \sum_n \frac{u_n(x_s)\, u_n(x)}{\omega_n^2 - \omega^2}}$$

To show this work in reverse — follow pp. 21–22 in 1982 notes

Just do mode sum followed by wave sum. $g(x|x_s, t)$

Lecture 3 :

Finish off derivation of $g(x|x_s, t)$ and $g(x|x_s, \omega)$

Mode sum to ray sum — for a uniform string.

Then go back to N-degree system

$$H\underline{q} = \omega^2 \underline{q} \qquad H = T^{-1} V$$

$$\langle \underline{q}, \underline{q}' \rangle = \underline{q}^T T \underline{q}$$

$$\omega^2 = \frac{\langle \underline{q}, H\underline{q} \rangle}{\langle \underline{q}, \underline{q} \rangle} = \frac{\underline{q}^T V \underline{q}}{\underline{q}^T T \underline{q}} = \frac{PE}{KE}$$

$$\omega^2 = \frac{\mathcal{V}}{\mathcal{T}}$$

Rayleigh's principle

$$\delta\omega^2 =$$

$$\delta\left(\frac{\mathcal{V}}{\mathcal{T}}\right) = \frac{\delta\mathcal{V}}{\mathcal{T}} - \frac{\mathcal{V}}{\mathcal{T}^2}\delta\mathcal{T} = \frac{\delta\mathcal{V} - \omega^2 \delta\mathcal{T}}{\mathcal{T}}$$

$$= \frac{1}{\mathcal{T}} \delta(\mathcal{V} - \omega^2 \mathcal{T})$$

$$\delta(\mathcal{V}-\omega^2\mathcal{T}) = \delta(\underline{q}^T V \underline{q} - \omega^2 \underline{q}^T T \underline{q})$$

$$= 2\delta\underline{q}^T (V\underline{q} - \omega^2 T\underline{q}) = 0 \quad \text{iff}$$

$$V\underline{q} = \omega^2 T \underline{q}$$

Now assume $\mathcal{V}=\mathcal{V}(\underline{q}, p)$, $\mathcal{T}=\mathcal{T}(\underline{q}, p)$

parameters

$$\mathcal{V}-\omega^2\mathcal{T}=0$$

$$\delta_{total}\mathcal{V} = 2\delta\underline{q}^T V\underline{q} + \underline{q}^T \delta V \underline{q}$$

$\delta_{total}\mathcal{T}$ = same

$$\mathcal{L} = \mathcal{V}-\omega^2\mathcal{T}$$

$$\mathcal{L}(\omega, \underline{q}; p) = \mathcal{L}(\omega+\delta\omega, \underline{q}+\delta\underline{q}; p+\delta p) = 0$$

$$\delta_{total}\mathcal{L} = 0$$

Elastic 3-d $\oplus$

To begin, ignore gravity

$$\oplus = \oplus_S + \oplus_F$$

$$\Sigma = \partial\oplus + \Sigma_{SS} + \Sigma_{FS}$$

Moho CMB
etc. etc.

$$\rho\, \partial_t^2 \underline{s} = \nabla \cdot \underline{\underline{T}}$$

$$\underline{\underline{T}} = \underline{\underline{\Gamma}} : \underline{\underline{\varepsilon}} . \qquad T_{ij} = \Gamma_{ijkl}\, \varepsilon_{kl}$$

$$\varepsilon = \frac{1}{2}\left[\nabla s + (\nabla s)^T\right]$$

$$\varepsilon_{kl} = \frac{1}{2}\left(\partial_k s_l + \partial_l s_k\right)$$

$$\Gamma_{ijkl} \overset{\text{trivial}}{=} \Gamma_{jikl} \overset{\text{trivial}}{=} \Gamma_{ijlk} \overset{\text{non-trivial (thermodynamic)}}{=} \Gamma_{klij}$$

b.c. $\quad \hat{n} \cdot \underline{\underline{T}} = \underline{0}$ on $\partial\oplus$ $\qquad$ dynamic

$\qquad [\hat{n} \cdot \underline{\underline{T}}]^+_- = \underline{0}$ on Σ_{SS}

$$[\hat{n} \cdot \underline{\underline{T}}]^+_- = \hat{n}\,[\hat{n} \cdot \underline{\underline{T}} \cdot \hat{n}]^+_- = \underline{0} \quad \text{on } \Sigma_{FS}$$

kinematic: $\quad [\hat{n} \cdot \underline{s}]^+_- = 0$ on Σ_{FS}

$\qquad [\underline{s}]^+_- = \underline{0}$ on Σ_{SS}

Conservation of energy:

$$\int_{\oplus} \partial_t s \, (\rho \partial_t^2 \underline{s} - \nabla \cdot T) \, dV$$

first term $\frac{d}{dt} \frac{1}{2} \int_{\oplus} \rho \, \partial_t s \cdot \partial_t s \, dV$ k.e.

do review of Gauss' theorem here first — then

potential energy:

$$-\int_{\oplus} \partial_t^2 s_j \, \partial_i \overset{T_{ij}}{\cancel{(\Gamma_{ijkl} \partial_k s_l)}} \, dV$$

$$= \int_{\Sigma} \underbrace{[\, n_i T_{ij} \, \partial_t^2 s_j \,]^{\pm}}_{0 \text{ by b.c.}} \, dA$$

check all 3 cases

$$+ \int_{\oplus} \partial_t^2 (\partial_i s_j) \, \Gamma_{ijkl} \, \partial_k s_l \, dV$$

$$= \frac{d}{dt} \int_{\oplus} \frac{1}{2} \partial_i s_j \, \Gamma_{ijkl} \, \partial_k s_l \, dV$$

this because $\Gamma_{ijkl} = \Gamma_{klij}$

$$= \frac{d}{dt} \frac{1}{2} \int_{\oplus} \varepsilon_{ij} \, \Gamma_{ijkl} \, \varepsilon_{kl} \, dV$$

$$= \frac{d}{dt} \frac{1}{2} \int \underbrace{\varepsilon : \underline{\Gamma} : \varepsilon}_{\text{p.e. density}} \, dV$$

$$\frac{d}{dt}(T+V) = 0$$

$$T = \frac{1}{2}\int_{\oplus} \rho\, |\partial_t \underline{s}|^2\, dV$$

$$V = \frac{1}{2}\int_{\oplus} \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{\Gamma}}}} : \underline{\underline{\varepsilon}}\, dV$$

derive local energy conservation law here — gives expression for flux of energy

Normal mode solutions — why do we look for these, by the way?

answer : ρ and $\underline{\underline{\underline{\underline{\Gamma}}}}$ are ind. of time

example : $\rho(z)$, $\underline{\underline{\underline{\underline{\Gamma}}}}(z)$ only

then $\underline{s} \sim e^{i(kx+ly)}$ harmonic in x & y.

$$\underline{s}(\underline{x},t) = \underline{s}(\underline{x})\, e^{i\omega t}$$

$$\partial_t^2 \to -\omega^2$$

$$-\rho\omega^2 \underline{s} = \nabla\cdot(\underline{\underline{\underline{\underline{\Gamma}}}} : \underline{\underline{\varepsilon}})$$

$$\rho H \underline{s} = -\nabla\cdot\underline{\underline{T}} = -\nabla\cdot(\underline{\underline{\underline{\underline{\Gamma}}}} : \underline{\underline{\varepsilon}})$$

$$H\underline{s} = \omega^2 \underline{s}$$

Define $\langle \underline{s}, \underline{s}' \rangle = \int_{\oplus} \rho\, \underline{s}\cdot\underline{s}'\, dV$

Then $\langle \underline{s}, H\underline{s}' \rangle = \langle H\underline{s}, \underline{s}' \rangle = \langle \underline{s}', H\underline{s} \rangle$

Proof: same ideas as cons. of energy

$$\langle \underline{s}, H\underline{s}' \rangle = - \int_{\oplus} \underline{s} \cdot (\nabla \cdot \underline{T}') \, dV$$

$$s_j \, \partial_i T_{ij}'$$

$$= \int_{\Sigma} [n_i T_{ij}' s_j]^{\pm} + \int_{\oplus} T_{ij}' \partial_i s_j \, dV$$

$\hookrightarrow$ zero

$$= \int \Gamma_{ijkl} \, \partial_i s_j \, \partial_k s_l' \, dV$$

$$= \int \Gamma_{ijkl} \, \partial_i s_j' \, \partial_k s_l \, dV$$

go backwards now

$$= \langle \underline{s}', H\underline{s} \rangle$$

Know now that eigenfrequencies^2 <u>real</u> and eigenfunctions orthogonal

$$(\omega^2 - \omega'^2) \int_{\oplus} \rho \, \underline{s} \cdot \underline{s}' \, dV = 0$$

Rayleigh's principle

$$H\underline{s} = \omega^2 \underline{s}$$

$$\omega^2 = \frac{\langle \underline{s}, H\underline{s} \rangle}{\langle \underline{s}, \underline{s} \rangle}$$

$$\delta\omega^2 = \frac{\langle \delta s, Hs \rangle + \langle s, H\delta s \rangle}{\langle s, s \rangle} - \frac{\langle s, Hs \rangle}{\langle s, s \rangle^2} \left[\langle \delta s, s \rangle + \langle s, \delta s \rangle \right]$$

$$= \frac{\langle \delta s, Hs \rangle + \langle s, H\delta s \rangle - \omega^2 \langle \delta s, s \rangle - \omega^2 \langle s, \delta s \rangle}{\langle s, s \rangle}$$

$$= \frac{2 \langle \delta s, Hs - \omega^2 s \rangle}{\langle s, s \rangle}$$

$$\omega^2 \int_\oplus \rho\, \underline{s} \cdot \underline{s}\, dV = \int_\oplus \varepsilon : \Gamma : \varepsilon$$

$$\omega^2 = \frac{\int_\oplus \varepsilon : \Gamma : \varepsilon \, dV}{\int_\oplus \rho\, \underline{s} \cdot \underline{s}\, dV} = \frac{PE}{KE}$$

Instead, can consider the stationary quantity to be

$$I = \frac{1}{2}\omega^2 \langle s, s \rangle - \frac{1}{2} \langle s, Hs \rangle$$

$$\delta I = \langle \delta s, \omega^2 s - Hs \rangle$$

$$I = \omega^2 \int_{\oplus} \rho\, s \cdot s \, dV - \int_{\oplus} \varepsilon : \Gamma : \varepsilon \, dV$$

$$\delta I = \int_{\oplus} \delta s \cdot (\omega^2 \rho \underline{s} + \nabla \cdot \underline{\underline{T}}) \, dV$$

$$+ \int_{\Sigma} [\delta s \cdot (\hat{n} \cdot \underline{\underline{T}})]_{-}^{+} \, dA = 0$$

dynamic

gives <u>equs</u> <u>AND</u> b.c.

Special case — isotropic

$$\Gamma_{ijkl} = (\kappa - \tfrac{2}{3}\mu)\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\delta_{jl} + \delta_{il}\,\delta_{jk})$$

$$T_{ij} = (\kappa - \tfrac{2}{3}\mu)(\nabla \cdot s)\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$$

$$= \kappa\, (\nabla \cdot s)\, \delta_{ij} + 2\mu\, d_{ij}$$

$$d = \varepsilon - \tfrac{1}{3}(\operatorname{tr}\varepsilon)\, I \qquad \text{deviatoric strain}$$

$$\operatorname{tr} d = 0$$

$$\varepsilon = \tfrac{1}{3} (\operatorname{tr}\varepsilon) I + d \quad , \quad \operatorname{tr} d = 0$$

$$T = \underbrace{\kappa (\nabla \cdot s) I}_{\text{isotropic}} + \underbrace{2\mu d}_{\text{deviatoric}}$$

elastic PE $= C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}$

$$= \underbrace{\kappa\,(\nabla\cdot s)^2}_{\text{bulk or comp energy}} + \underbrace{2\mu\,\underline{\underline{d}}:\underline{\underline{d}}}_{\text{shear energy}}$$

$d_{ij}\,d_{ij}$

$$I = \frac{1}{2}\int_{\oplus} [\omega^2 \rho\, \underline{s}\cdot\underline{s} - \kappa(\nabla\cdot\underline{s})^2 - 2\mu\,(\underline{\underline{d}}:\underline{\underline{d}})]\,dV$$

Stability = all elastic material must have

Γ pos. def $\Rightarrow$

$\varepsilon : \Gamma : \varepsilon > 0$ for all ε

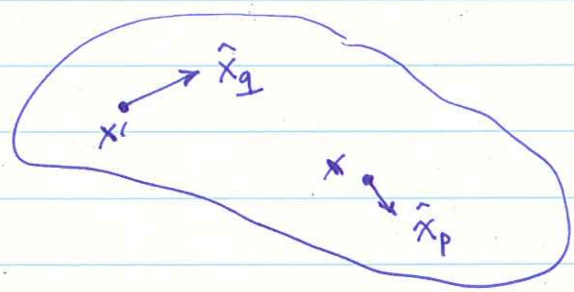
Every deformation increases potential energy, i.e., requires work.

Isotropic $\quad \kappa\,(\nabla\cdot s)^2 + 2\mu\,(d:d) > 0$ for every ε

$\Rightarrow \quad \kappa > 0\,, \quad \mu > 0.$

Excitation — same as for a string

Define $G_{pq}(x, x'; t) = \hat{x}_p$ comp of response at x, t due to unit impulse in $\hat{x}_q$ direction at $x', 0$.



modes $\omega_k, s_k(x)$

$$\langle s_k, s_{k'} \rangle = \int_{\oplus} \rho\, s_k \cdot s_{k'}\, dV = \delta_{kk'}$$

normalized

$$\rho(\partial_t^2 G + HG) = I\,\delta(x - x')\,\delta(t)$$

or
$$\rho(\partial_t^2 G + HG) = 0$$

$$G(0) = 0 \qquad \rho\dot{G}(0) = I\,\delta(x - x')$$

$$G(t) = \sum_k s_k(x)\left[a_k(x')\cos\omega_k t + b_k(x')\sin\omega_k t\right]$$

$$\sum_k s_k b_k = 0$$

$$\sum_k \omega_k s_k a_k = \rho^{-1} I\,\delta(x - x')$$

$$b_k = 0 \qquad a_k = \omega_k^{-1} s_k(x')$$

$$\boxed{G(x,x';t) = \sum_k \omega_k^{-1} s_k(x)\, s_k(x')\, \sin\omega_k t}$$

$$\boxed{G(x,x';t) = G^T(x',x;t) \quad \text{reciprocity}}$$

[Could do response to a transient force next.

Pert theory :

$$I = \tfrac{1}{2}\left[\omega^2 \langle s,s\rangle - \langle s, Hs\rangle\right]$$

$$\delta I_{total} = 0$$

$$\delta\omega^2 \langle s,s\rangle = \langle s, \delta H s\rangle$$

$$\boxed{\delta\omega^2 = \frac{\langle s, \delta H s\rangle}{\langle s,s\rangle}}$$

$$I = \frac{1}{2}\left[\omega^2 \int_\oplus \delta\rho\, s\cdot s\, dV - \int_\oplus \varepsilon : \underline{\Gamma} : \varepsilon\, dV\right]$$

$$\delta\omega^2 = \frac{\int_\oplus \left[\varepsilon : \delta\Gamma : \varepsilon - \omega^2 \delta\rho\, s\cdot s\right] dV}{\int_\oplus \rho\, s\cdot s\, dV}$$

Gives change in ω^2 due to change in $\oplus$ model

$$\Gamma_{ijkl} \to \Gamma_{ijkl} + \delta\Gamma_{ijkl}$$

$$\rho \to \rho + \delta\rho$$

Isotropic

$$\delta\omega^2 = \frac{\int_\oplus [\delta\kappa\,(\nabla\cdot s)^2 + 2\,\delta\mu\,(d:d) - \omega^2\,\delta\rho\, s\cdot s]\,dV}{\int_\oplus \rho\, s\cdot s\, dV}$$

Due to changes $\delta\kappa$, $\delta\mu$, $\delta\rho$

$\boxed{\text{Energy flux}}$

$$\rho \partial_t^2 s = \nabla \cdot T$$

$$\rho\, \partial_t s \cdot \partial_t^2 s = \frac{d}{dt}\left(\frac{1}{2}\rho\, \partial_t s \cdot \partial_t s\right)$$

$$\partial_t s \cdot \nabla \cdot (\Gamma : \varepsilon)$$

$$= \partial_t s_j\, \partial_i (\Gamma_{ijkl}\, \varepsilon_{kl})$$

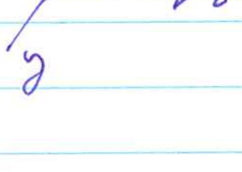
$$= \partial_i (\partial_t s_j\, \Gamma_{ijkl}\, \varepsilon_{kl})$$

$$- \Gamma_{ijkl}\, \partial_t \varepsilon_{ij}\, \varepsilon_{kl}$$

$$\boxed{\frac{dE}{dt} + \nabla \cdot \underline{K} = 0}$$

$$\boxed{\begin{aligned} E &= \frac{1}{2}\rho(\partial_t s)^2 + \frac{1}{2}(\varepsilon : \Gamma : \varepsilon) \\ \underline{K} &= -\partial_t s \cdot \underline{\underline{T}} \end{aligned}}$$

Plane P wave



$$s_x = s_y = 0$$

$$s_z = A \sin(kx - \omega t)$$

$$\left(\kappa - \tfrac{2}{3}\mu\right)$$

$$T_{ij} = (\nabla \cdot s)\,\delta_{ij} + 2\mu\left(\partial_i s_j + \partial_j s_i\right)$$

$$T_{zz} = \cancel{\left(\kappa + 2\mu\right)\partial_z s_z}$$

$$= \left(\kappa + \tfrac{4}{3}\mu\right)\partial_z s_z$$

$$\cancel{K_z = -i\omega A\left(\kappa + \tfrac{4}{3}\mu\right)(-ikA)\,e^{i(\omega t - kx)}}$$

$$K_z = -\omega A\left(-\cos(kx - \omega t)\right)$$

$$\times\, k\left(\kappa + \tfrac{4}{3}\mu\right) A \cos(kx - \omega t)$$

$$= \omega k\left(\kappa + \tfrac{4}{3}\mu\right) A^2 \cos^2(kx - \omega t)$$

$$\kappa + \tfrac{4}{3}\mu = \rho\alpha^2$$

$$\omega k = \frac{\omega^2}{\alpha}$$

$$K_z = \rho\omega^2\alpha A^2 \cos^2(kx - \omega t)$$

$$\langle K_z \rangle = \tfrac{1}{2}\rho\omega^2\alpha A^2$$

unit check $\quad \dfrac{m}{l^3}\dfrac{l^3}{t^3} = \underbrace{m\dfrac{l^2}{t^2}}_{E}\ \dfrac{1}{l^2 t}$

$\dfrac{\text{energy}}{m^2\ \text{sec}}$

Lecture #4

also b.c. on $\Sigma = \Sigma_{FF} + \Sigma_{SS} + \partial\oplus$

$$\rho \partial_t^2 \underline{s} = \nabla \cdot \underline{\underline{T}}$$

$$T = \Gamma : \varepsilon \qquad T_{ij} = \Gamma_{ijkl}\, \varepsilon_{kl}$$

Normal modes $\underline{s}(\underline{x},t) = \underline{s}(\underline{x})\, e^{i\omega t}$

$$\partial_t^2 = -\omega^2$$

$H\underline{s} = \omega^2 \underline{s}$ — eigenvalue problem

$$H\underline{s} = -\frac{1}{\rho} \nabla \cdot \underline{\underline{T}} = -\frac{1}{\rho} \nabla \cdot (\underline{\Gamma} : \varepsilon)$$

KE inner product

$$\langle \underline{s}, \underline{s}' \rangle = \int_{\oplus} \rho\, \underline{s} \cdot \underline{s}'\, dV$$

Hermitian

$$\langle \underline{s}, H\underline{s}' \rangle = \langle H\underline{s}, \underline{s}' \rangle$$

Concept intimately involves b.c.

Proof — repeat cons. of energy arguments

$$\langle \underline{s}, H\underline{s}' \rangle = \int_{\oplus} \rho s_j \left(-\frac{1}{\rho} \partial_i T_{ij}' \right) dV$$

$$= -\int_{\oplus} s_j \partial_i T_{ij}' \, dV$$

vanishes on all 3 bdry types

$$= \int_{\Sigma} [n_i T_{ij}' s_j]^{\pm} \, dA$$

$$+ \int_{\oplus} T_{ij}' \partial_i s_j \, dV$$

$$= \int_V \Gamma_{ijkl} \varepsilon_{kl}' \varepsilon_{ij} \, dV$$

$$\Gamma_{ijkl} = \Gamma_{klij}$$

$$= \int_V \Gamma_{ijkl} \varepsilon_{kl} \varepsilon_{ij}' \, dV$$

now go backwards

$$= \langle \underline{s}', H\underline{s} \rangle \quad \text{QED}$$

Now know that all ω_n^2 real

all $s_n(x)$ orthogonal

Modes $\omega_n^2, \quad \underline{s}_n(\underline{x}), \quad n = 1, 2, \ldots$

$$\langle \underline{s}_n, \underline{s}_{n'} \rangle = \int_{\oplus} \rho \, \underline{s}_n \cdot \underline{s}_{n'} \, dV$$
$$= \delta_{nn'}$$

Also true that all $\omega_n^2 > 0$ (in absence of gravity)

$$H \underline{s}_n = \omega_n^2 \underline{s}_n$$

$$\omega_n^2 = \frac{\langle s_n, H s_n \rangle}{\langle s_n, s_n \rangle}$$

$$= \frac{\int_{\oplus} \underline{\underline{\varepsilon}}_n : \underline{\underline{\underline{\underline{\Gamma}}}} : \underline{\underline{\varepsilon}}_n \, dV}{\int_{\oplus} \rho \, \underline{s}_n \cdot \underline{s}_n \, dV}$$

but $\underline{\underline{\underline{\underline{\Gamma}}}}$ is positive definite $\Rightarrow \omega_n^2 > 0$

Every deformation must store positive PE.

Isotropic

$$\omega_n^2 = \frac{\int_{\oplus} \kappa (\nabla \cdot s_n)^2 \, dV + \int_{\oplus} 2\mu \, (\underline{\underline{d}}_n : \underline{\underline{d}}_n) \, dV}{\int_{\oplus} \rho \, \underline{s}_n \cdot \underline{s}_n \, dV}$$

normalization $\Rightarrow$ denominator $= 1$

$$f_\kappa = \omega_n^{-2} \int_{\oplus} \kappa (\nabla \cdot \underline{s}_n)^2 \, dV$$

$$f_\mu = \omega_n^{-2} \int_{\oplus} 2\mu \, (\underline{\underline{d}}_n : \underline{\underline{d}}_n) \, dV$$

$$f_\kappa + f_\mu = 1$$ fractional compressional and shear PE.

Rayleigh's principle — did not do for N-degree system or violin string, but could have.

Every $H\underline{s} = \omega^2 \underline{s}$ eigenvalue problem has an associated variational principle.

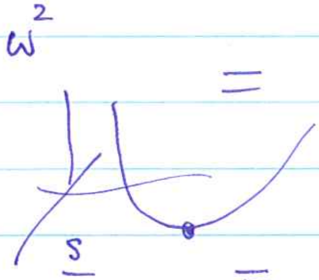
$$H\underline{s} = \omega^2 \underline{s}$$

$$\omega^2 = \frac{\langle \underline{s}, H\underline{s} \rangle}{\langle \underline{s}, \underline{s} \rangle} \leftarrow \text{functional of } \underline{s}$$

$$\delta\omega^2 = \omega^2(\underline{s} + \underline{\delta s}) - \omega^2(\underline{s})$$

$$= \frac{\langle \delta s, Hs \rangle + \langle s, H\delta s \rangle}{\langle s, s \rangle} - \frac{\langle s, Hs \rangle}{\langle s, s \rangle^2} \Big[\langle \delta s, s \rangle + \langle s, \delta s \rangle \Big]$$

$$= \frac{\langle \delta s, Hs \rangle + \langle s, H\delta s \rangle - \omega^2 \langle \delta s, s \rangle - \omega^2 \langle s, \delta s \rangle}{\langle s, s \rangle}$$



$$= \frac{2 \langle \delta s, Hs - \omega^2 s \rangle}{\langle s, s \rangle} \qquad \text{Hermitian}$$

$= 0$ for arbitrary δs iff $H\underline{s} - \omega^2 \underline{s} = 0$

$$\omega^2(s) = \frac{PE}{KE} = \frac{\int_\oplus \underline{\varepsilon} : \underline{\underline{\Gamma}} : \underline{\varepsilon} \, dV}{\int_\oplus \rho \underline{s} \cdot \underline{s}}$$

Can instead regard $I = \frac{1}{2}\omega^2 \langle s, s \rangle - \frac{1}{2}\langle s, Hs \rangle$ as stationary functional — for fixed ω

$$\delta I = I(s + \delta s) - I(s)$$

$$= \langle \delta s, \omega^2 s - Hs \rangle$$

$$I = \frac{1}{2}\omega^2 \int_\oplus \rho s \cdot s \, dV - \frac{1}{2}\int_\oplus \underline{\varepsilon} : \underline{\underline{\Gamma}} : \underline{\varepsilon} \, dV$$

$$= \omega^2 KE - PE$$

gives eqns & BC.

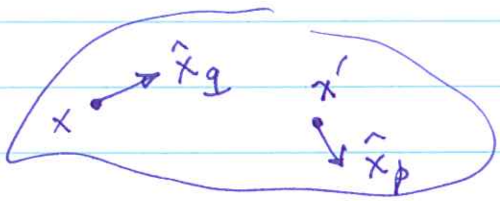
$$\delta I = \int_\oplus \delta s \cdot (\omega^2 \rho \underline{s} + \nabla \cdot \underline{T}) \, dV$$

$$+ \int_\Sigma [\delta \underline{s} \cdot (\hat{n} \cdot \underline{T})]^+_- \, dA$$

Isotropic :

$$I = \frac{1}{2}\int_{\oplus} [\rho\omega^2 s\cdot s - \kappa(\nabla\cdot s)^2 - 2\mu(\underline{\underline{d}}:\underline{\underline{d}})]\, dV$$

Excitation — impulse response — Green tensor



$G_{pq}(x|x', t) = \hat{x}_p$ component of response at x, t due to unit impulse in $\hat{x}_q$ direction at $x', 0$.

$$\rho(\partial_t^2 \underline{\underline{G}} + H\underline{\underline{G}}) = \underbrace{\underline{\underline{I}}\,\delta(x-x')\,\delta(t)}_{\text{impulse — homog IC}}$$

or

$$\rho(\partial_t^2 \underline{\underline{G}} + H\underline{\underline{G}}) = \underline{\underline{0}}$$

$$\underline{\underline{G}}(x|x', 0) = 0$$

$$\partial_t \underline{\underline{G}}(x|x', 0) = \frac{1}{\rho} I\, \delta(x-x')$$

$$G(t) = \sum_k s_k(x)\,[a_k(x')\cos\omega_k t + b_k(x')\sin\omega_k t]$$

$$\sum_k s_k a_k = \underline{\underline{0}}, \quad \sum_k \omega_k s_k b_k = \rho^{-1} I\, \delta(x-x')$$

$$a_k = 0, \quad b_k = \omega_k^{-1} s_k(x')$$

$$\boxed{G(x|x', t) = \sum_k \omega_k^{-1} s_k(x)\, s_k(x')\sin\omega_k t}$$

Reciprocity : $\boxed{G(x|x',t) = G^T(x'|x,t)}$

Discuss physical interpretation.

VERY general result — general $\oplus$ model $\rho(x)$, $\underline{\underline{\Gamma}}(x) \leftarrow$ 21 coefficients

Homework — Eimei's paradox

Lecture #6

(1) followup on Eisner's paradox

homog medium — pressure impulse response

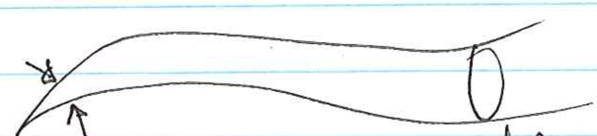
$$\nabla^2 g - \frac{1}{c^2}\partial_t^2 g = -\rho\,\delta(t)\,\delta(\underline{x}-\underline{x}')$$

boom

$$g(\underline{x}|\underline{x}',t) = \rho\frac{\delta(t - R/c)}{R}$$

$$R = \|\underline{x} - \underline{x}'\|.$$

more generally $\quad R \rightarrow \mathcal{R} = \sqrt{\dfrac{dA}{d\Omega}}$



i.e., $\quad$ response $\sim \dfrac{1}{\mathcal{R}} \sim \sqrt{\dfrac{d\Omega}{dA}}$

In Eisner case $\quad dA \rightarrow 0 \Rightarrow$
response $\rightarrow \infty$

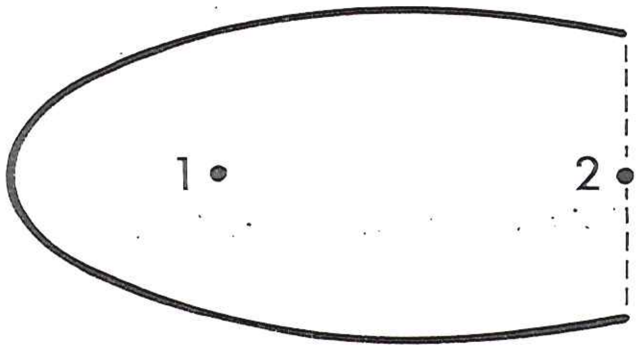


FIG. 1. Eisner's truncated ellipsoidal geometry leading to an apparent failure of the reciprocity principle. Source and receiver are located at the focal points 1 and 2.

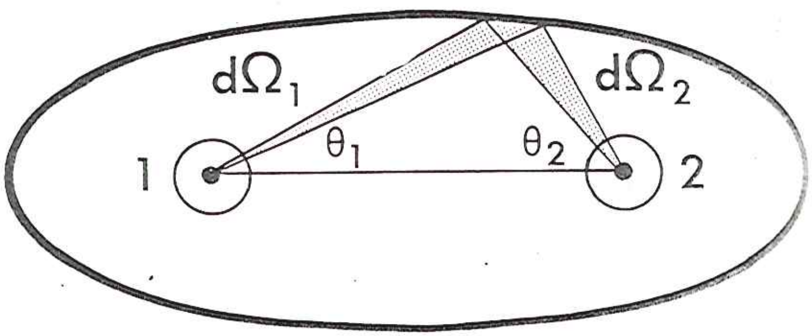


FIG. 2. Geometry of a complete ellipsoid showing the ray takeoff angles θ_1 and θ_2 and the elementary solid angles $d\Omega_1$ and $d\Omega_2$. Focal spheres of unit radius are centered on each focus; a wavefront departing the focal sphere at 1 converges after reflection onto that at 2, and vice versa. Note the crossing of adjacent rays.

If the amplitude on the focal sphere at 1 is unity, that on the focal sphere at 2 is

$$A(\theta_2) = \left|\frac{d\Omega_1}{d\Omega_2}\right|^{1/2} = \left|\frac{\sin\theta_1 d\theta_1}{\sin\theta_2 d\theta_2}\right|^{1/2} \tag{1}$$

by conservation of energy. The amplitude distribution produced on the focal sphere at 1 by an isotropic source at 2 is, likewise, $A(\theta_1)$. Using elementary analytical geometry it can be shown that the takeoff angles θ_1 and θ_2 satisfy

$$(1 - 2\varepsilon\cos\theta_1 + \varepsilon^2)(1 - 2\varepsilon\cos\theta_2 + \varepsilon^2) = (1 - \varepsilon^2)^2, \tag{2}$$

where ε is the eccentricity of the ellipsoid. The semimajor and semiminor axes a and b are related to the eccentricity ε by $b = a(1 - \varepsilon^2)^{1/2}$. Upon differentiating equation (2), we find that the amplitude distribution function is given by

$$A(\theta) = \frac{1 - \varepsilon^2}{1 - 2\varepsilon\cos\theta + \varepsilon^2}, \tag{3}$$

where θ denotes either θ_1 or θ_2. This is plotted for various values of ε in Figure 3.

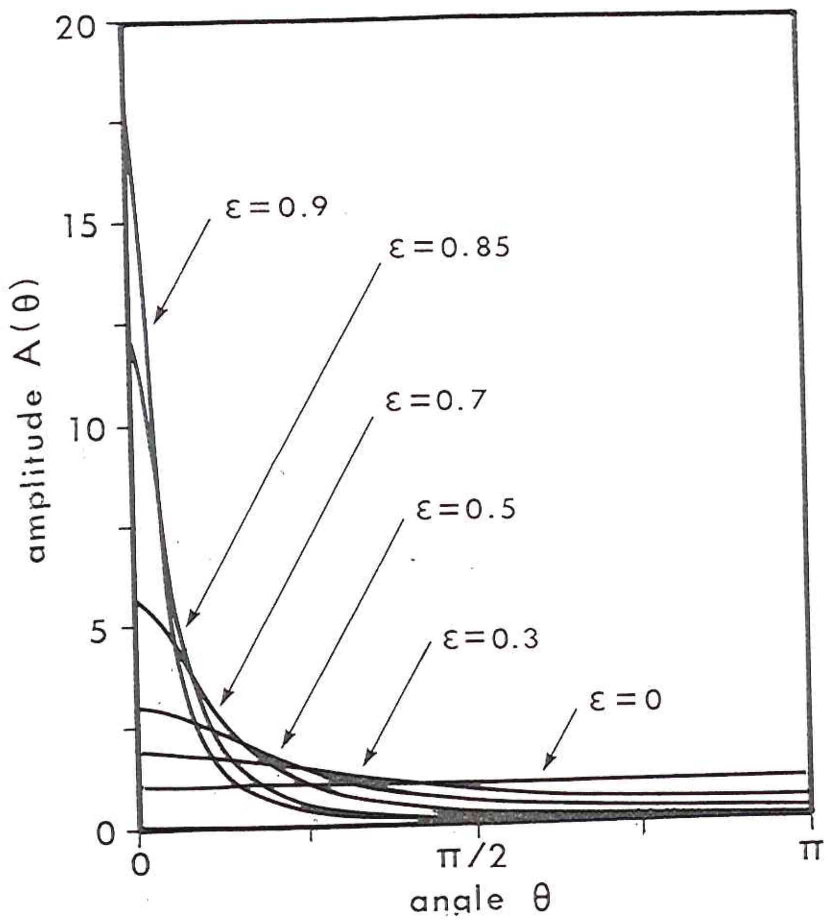


FIG. 3. Amplitude distribution on the incoming focal sphere due to an isotropic source at the other focus. Note the strong forward focusing for large values of the eccentricity ε.

Debye's theory stipulates that amplitude of the reflected wave arriving at the focus is integral of the incoming amplitude distribution over the f sphere:

$$U = 2\pi \int_0^{\pi} A(\theta) \sin \theta \, d\theta.$$

This simple result is valid only right at the focus; the diffrac pattern in the vicinity of the focus is wavelength-dependent quite complicated. From equations (3) and (4) the focal an tude in the case of a complete ellipsoid is

$$U = \frac{2\pi(1 - \varepsilon^2)}{\varepsilon} \ln \left(\frac{1 + \varepsilon}{1 - \varepsilon} \right).$$

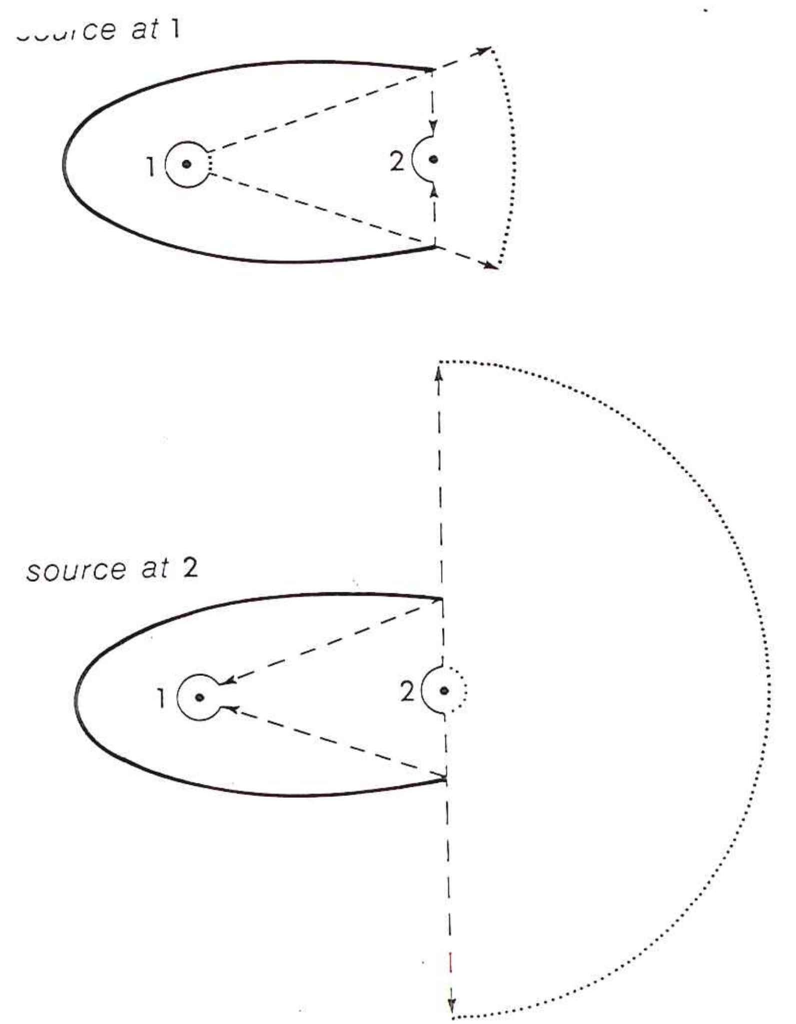


FIG. 4. The solid half of the wavefront leaving 2 is focused by reflection onto 1 while the dotted half continues its spherical expansion. A smaller dotted portion of the wavefront leaving 1 is unreflected.

Schemati snapshots of the wavefronts in Eisner's problem are shown in Figure 4. In this case only half the focal sphere at 2 is illuminated by a source at 1 and the resulting amplitude is

$$U_2 = 2\pi \int_0^{\pi/2} A(\theta) \sin d\theta. \tag{6}$$

The corresponding shadow boundary on the focal sphere at 1 due to a source at 2 occurs at the maximum of the integrand

$$\theta_1 = \arccos\left(\frac{2\varepsilon}{1+\varepsilon^2}\right) \tag{7}$$

This follows from equation (2) with $\theta_2 = \pi/2$. The amplitu 1 is therefore

$$U_1 = 2\pi \int_{\arccos[2\varepsilon/(1+\varepsilon^2)]}^{\pi} A(\theta) \sin \theta \, d\theta. \tag{8}$$

Evaluating the above integrals, we find that

$$U_1 = U_2 = \frac{\pi(1-\varepsilon^2)}{\varepsilon} \ln \left[\frac{1+\varepsilon^2}{(1-\varepsilon)^2}\right] \tag{9}$$

which is consistent with the reciprocity principle, as of course it must be. The nature of the equality is illustrated in Figure 5 for the case $\varepsilon = 0.5$.

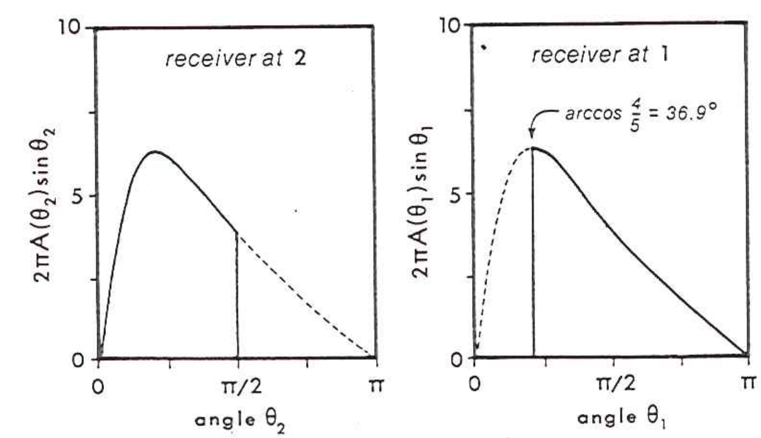


FIG. 5. Comparison of the Debye integrands $2\pi A(\theta) \sin \theta$ for an ellipsoid of eccentricity $\varepsilon = 0.5$.

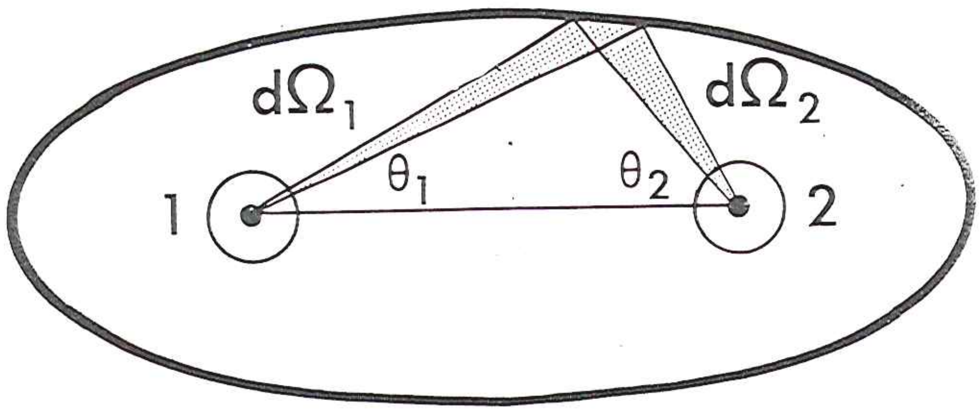


FIG. 2. Geometry of a complete ellipsoid showing the ray takeoff angles θ_1 and θ_2 and the elementary solid angles $d\Omega_1$ and $d\Omega_2$. Focal spheres of unit radius are centered on each focus; a wavefront departing the focal sphere at 1 converges after reflection onto that at 2, and vice versa. Note the crossing of adjacent rays.

More generally it is easy to see there is no violation of reciprocity no matter what portion or portions of the ellipsoid may have been removed. For a source at 1 the incremental amplitude produced at 2 by reflection off an arbitrary small patch on the ellipsoid is, from Debye's theory, given by

$$dU_2 = |d\Omega_1/d\Omega_2|^{1/2} d\Omega_2 = |d\Omega_1 d\Omega_2|^{1/2}. \tag{11}$$

The corresponding signal produced at 1 by reflection off the same patch has an amplitude

$$dU_1 = |d\Omega_2/d\Omega_1|^{1/2} d\Omega_1 = |d\Omega_1 d\Omega_2|^{1/2}. \tag{12}$$

Since $dU_1 = dU_2$ for every patch and different patches are independent in the geometrical optics limit, reciprocity is always guaranteed. This argument is a corrected version of Eisner's own second "objection" to his proposed "counterexample."

Applications of Rayleigh's principle

(1) Rayleigh – Ritz method – page 12 of 1982 notes

(2) perturbation theory.

$$I(\underline{s}) = \tfrac{1}{2}\left[\omega^2\langle \underline{s},\underline{s}\rangle - \langle \underline{s}, H\underline{s}\rangle\right] = 0$$

when evaluated at a stationary point (i.e., a mode)

~~Regard as~~ $\cancel{I(\underline{s}, \oplus)}$ ~~$\omega, \underline{s}, \oplus$~~ ~~$\rho, \Gamma_{ijkl}$~~

$\cancel{\delta I_{total} = I(\underline{s} + \delta\underline{s}, \oplus + \delta\oplus) - I(\underline{s}, \oplus)}$

~~where this is pert due to~~

Regard as $I(\omega^2, \underline{s}, \oplus)$

$\rho(\underline{x}),\ \Gamma_{ijkl}(\underline{x})$

$$\oplus \to \oplus + \delta\oplus \;\Rightarrow\; \omega^2 \to \omega^2 + \delta\omega^2,\quad \underline{s} \to \underline{s} + \delta\underline{s}$$

$$I(\omega^2 + \delta\omega^2, \underline{s} + \delta\underline{s}, \oplus + \delta\oplus) = I(\omega^2, \underline{s}, \oplus) = 0$$

$$\delta I_{total} = I(\omega^2 + \delta\omega^2, \underline{s} + \delta\underline{s}, \oplus + \delta\oplus) - I(\omega^2, \underline{s}, \oplus)$$

$$= \langle \delta \underline{s}, \overbrace{\omega^2 \underline{s} - H\underline{s}}^{\text{zero by virtue of R. principle}} \rangle$$

$$+ \frac{1}{2}\delta\omega^2 \langle \underline{s}, \underline{s} \rangle - \frac{1}{2}\langle \underline{s}, \delta H \underline{s} \rangle$$

$$\boxed{\delta\omega^2 = \frac{\langle \underline{s}, \delta H \underline{s} \rangle}{\langle \underline{s}, \underline{s} \rangle}} \text{ — gen'l formula}$$

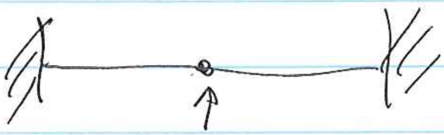
$$I = \frac{1}{2}\omega^2 \int_\oplus \rho\, \underline{s}\cdot\underline{s}\, dV - \frac{1}{2}\int_\oplus \Gamma_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}\, dV$$

$$\delta\omega^2 = \frac{\int_\oplus [\delta\Gamma_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} - \omega^2 \delta\rho\, \underline{s}\cdot\underline{s}]\, dV}{\int_\oplus \rho\, \underline{s}\cdot\underline{s}\, dV}$$

$$= \frac{\int_\oplus [\delta\kappa (\nabla\cdot\underline{s})^2 + 2\delta\mu\, (\underline{\underline{d}} : \underline{\underline{d}}) - \omega^2 \delta\rho\, \underline{s}\cdot\underline{s}]\, dV}{\int_\oplus \rho\, \underline{s}\cdot\underline{s}\, dV}$$

String:

$$\frac{\delta\omega^2}{\omega^2} = -\frac{\int_0^L \delta\rho(x)\, u^2(x)\, dx}{\int_0^L \rho(x)\, u^2(x)\, dx}$$



bead mass m

$$\delta\rho(x) = m\,\delta\left(x-\frac{L}{2}\right)$$

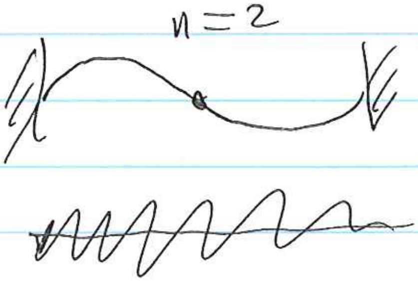
$$u_n(x) \sim \sin\frac{n\pi x}{L}$$

$$\frac{\delta\omega_n^2}{\omega_n^2} = -\frac{m\sin^2\frac{n\pi}{2}}{\rho L/2}$$

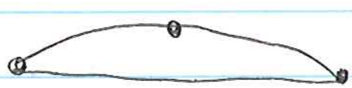
$$\frac{\delta\omega_n^2}{\omega_n^2} = -\left(\frac{2m}{\rho L}\right)\sin^2\frac{n\pi}{2}$$

$$= \begin{cases} -\frac{2m}{\rho L} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$n=2$



$n=1$



frequency graver — makes sense.

3/27/00 — first class after break

$-\rho\omega^2 \underline{s} = \nabla\cdot\underline{\underline{T}} + \text{gravity}$

$\underline{\underline{T}} = \kappa(\nabla\cdot\underline{s})\,\underline{\underline{I}} + 2\mu\underline{\underline{d}}$

$-\rho\omega^2 \underline{s} = (\kappa + \frac{1}{3}\mu)\nabla(\nabla\cdot\underline{s}) - \mu\nabla^2\underline{s}$

$- (\dot{\kappa} - \frac{2}{3}\dot{\mu})(\nabla\cdot\underline{s})\,\hat{r}$

$+ 2\dot{\mu}\,[\partial_r \underline{s} + \frac{1}{2}\hat{r}\times(\nabla\times\underline{s})]$

$+ \text{gravity}$

$$\underline{s} = \underbrace{U\,\underline{P}_{lm} + V\,\underline{B}_{lm}}_{\text{spheroidal}} + \underbrace{W\,\underline{C}_{lm}}_{\text{toroidal}}$$

decoupled spheroidal (4th or 6th order)
and toroidal (second order)

three integer quantum numbers

$n,\ l,\ m$

toroidal : $\ {}_n\underline{s}^T_{lm} = {}_nW_l(r)\,\underline{C}_{lm}(\theta,\phi) \qquad {}_n\omega_l^T$

spheroidal : $\ {}_n\underline{s}^S_{lm} = {}_nU_l(r)\,\underline{P}_{lm} + {}_nV_l(r)\,\underline{B}_{lm} \qquad {}_n\omega_l^S$

orthonormality

$$\int_0^a \rho \left({}_nU_l \, {}_{n'}U_{l'} + {}_nV_l \, {}_{n'}V_{l'} \right) r^2 \, dr = 1$$

$$\int_0^a \rho \, {}_nW_l \, {}_{n'}W_{l'} \, r^2 \, dr = 1$$

Now discuss toroidal modes
Start D&T page 280

Designation ${}_nT_l$

Class 4/6/2000

Back to Ying's question

1-d distributions on (1,1)

- $\langle f, \_ \rangle \to \langle f, \phi \rangle$

  $\uparrow \phi$

  written as $\langle f, \phi \rangle = \int_{-1}^{1} f \phi \, dV$ ← symbolic only

- differentiation $f^{(1)} = \frac{df}{dx}$ ← ordinary functions

$$\langle f^{(1)}, \phi \rangle \underset{\text{defn}}{=} - \langle f, \phi^{(1)} \rangle$$

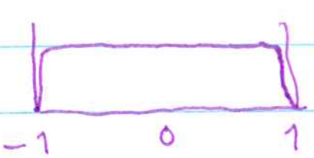
only differ. in this sense

$$\langle f^{(n)}, \phi \rangle = (-1)^n \langle f, \phi^{(n)} \rangle$$

- example — $f = \delta(x)$ ← delta at zero

  $\phi = 1$

note — can't be 1 since must vanish on ends $x = \pm 1$

ONE


$$\langle \delta, ONE \rangle = \int_{-1}^{1} \delta \; ONE \; dx = 1$$

$$\langle \delta', ONE \rangle = -\int_{-1}^{1} \delta \; ONE' \; dx = 0$$

Ying wanted $\langle \delta', ONE \rangle = \int_{-1}^{1} \delta' \; \cancel{ONE}^{\,1} \; dx$

$= \delta$

But limits are *fixed*

variable

- integration of a 1-d distribution $f^{(-1)} = \int_{-1}^{x} f \, dx$

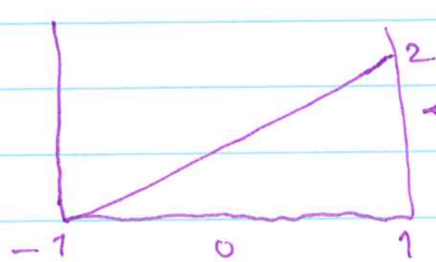
$$\langle f^{(-1)}, \phi \rangle = -\langle f, \phi^{(-1)} \rangle$$

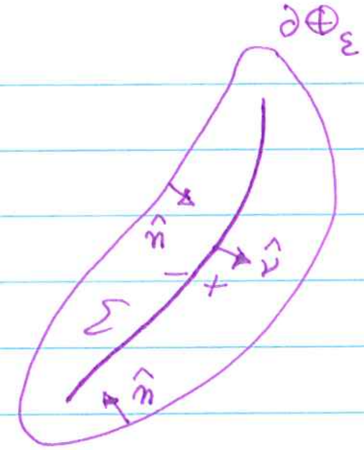
No – wait – this is *not* allowed since $\phi^{(-1)}$ is *not* in the space of test functions

if $\phi \in$ space then $\phi^{(n)} \in$ space

but this not true for $\phi^{(-1)}$

e.g. $ONE^{(-1)}$





$$\langle \nabla(\mathcal{D}f), \phi \rangle = -\langle \mathcal{D}f, \phi \rangle$$

$$= -\lim_{\varepsilon\to 0} \int_{\Theta-\Theta_\varepsilon} f \, \nabla\phi \, dV$$

$$= \lim_{\varepsilon\to 0} \left[\int_{\Theta-\Theta_\varepsilon} \nabla f \, \phi \, dV - \int_{\partial\Theta} \hat{n} f \phi \, dA \right]$$

$$= \int_{\Theta} \nabla f \, \phi \, dV + \int_{\Sigma} \hat{\nu} [f]^{\pm} \phi \, d\Sigma$$

$$= \langle \mathcal{D}(\nabla f), \phi \rangle + \langle \hat{\nu}[f]^{\pm} \delta_\Sigma, \phi \rangle$$

$$= \langle \mathcal{D}(\nabla f), \phi + \hat{\nu}[f]^{\pm} \delta_\Sigma, \phi \rangle$$

$$\boxed{\nabla(\mathcal{D}f) = \mathcal{D}(\nabla f) + \hat{\nu}[f]^{\pm} \delta_\Sigma}$$

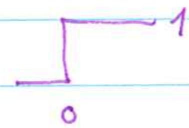
Class Tue 4/11/2000

~~[illegible] in homogeneous media~~

Point source model of an earthquake:

$$\mathbf{S} = \mathbf{M}\,\delta(\mathbf{x}-\mathbf{x}_s)\,H(t)$$

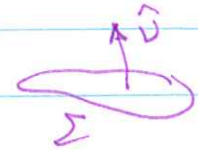
Heaviside step function



$$\mathbf{M} = \int_S \mathbf{S}_{final}\, dV \quad \text{volumetric source}$$

$$\mathbf{M} = \int_\Sigma \mu\,(\hat{\nu}\,\Delta\mathbf{s}_{final} + \Delta\mathbf{s}_{final}\,\hat{\nu})\, d\Sigma$$

fault source



Equivalent force

$$\mathbf{f} = -\nabla\cdot\mathbf{S} = -\mathbf{M}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_s)\,H(t)$$

Mode-sum Green tensor

$$\mathbf{G}(\mathbf{x}|\mathbf{x}_s, t) = \sum_k \omega_k^{-1}\,\mathbf{s}_k(\mathbf{x}_s)\,\mathbf{s}_k(\mathbf{x})\,\sin\omega_k t$$

$$\mathbf{G}(\mathbf{x}|\mathbf{x}_s, \omega) = \sum_k \frac{\mathbf{s}_k(\mathbf{x}_s)\,\mathbf{s}_k(\mathbf{x})}{\omega_k^2-\omega^2}$$

Response to force $f = -M \cdot \nabla \delta(x - x_s) H(t)$

$$s(x,\omega) = \underbrace{(i\omega)^{-1}}_{\text{FT of } H(t)} M : \nabla_s G^T(x|x_s, \omega)$$

$$s(x,t) = \sum_k \omega_k^{-2} M : \varepsilon_k(x_s)\, s_k(x)\, [1 - \cos\omega_k t]$$

Green tensor for ∞ homogeneous medium

Acoustic: wave eqn

$$\ddot{g} - \frac{1}{c^2}\nabla^2 g = \delta(x - x_s)\,\delta(t)$$

$$g(x|x_s, t) = \frac{\delta(t - R/c)}{4\pi c^2 R}$$

$$R = |x - x_s|$$

$\frac{1}{R}$ geometrical attenuation

$\delta(t - R/c)$ retarded time dependence

∞ elastic medium

$$\rho\, \partial_t^2 \underline{s} = \nabla \cdot \underline{\underline{T}} - \hat{x}_j\, \delta(x - x_s)\, \delta(t)$$

direction of applied force

$$\boxed{\begin{aligned} G_{ij}(x|x_s, t) &= \frac{1}{4\pi\rho\alpha^2}\, \gamma_i \gamma_j\, \frac{\delta(t - R/\alpha)}{R} \\ &+ \frac{1}{4\pi\rho\beta^2}\left(\delta_{ij} - \gamma_i\gamma_j\right) \frac{\delta(t - R/\beta)}{R} \\ &+ \frac{1}{4\pi\rho}\left(3\gamma_i\gamma_j - \delta_{ij}\right) \frac{1}{R^3} \int_{R/\alpha}^{R/\beta} \tau\, \delta(\tau - t)\, d\tau \end{aligned}}$$

$\gamma_j = \frac{x_j}{r}$ i.e. $\hat{r} = \gamma_1, \gamma_2, \gamma_3$

components of unit vector

$$G_{ij}(x|x_s, t) = G_{ji}(x_s|x, t)$$

reciprocity

3 terms:

far-field P wave
far-field S wave
near-field terms

far-field terms $\sim \frac{1}{R}$

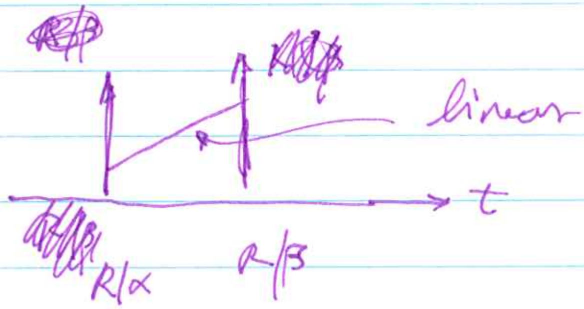
$\delta(t - R/c)$ dependence

Radiation patterns — AR Fig. 4.2

near-field term:

$$\int_{R/\alpha}^{R/\beta} \tau\,\delta(\tau - t)\,d\tau = \begin{cases} 0 & t \le R/\alpha \\ t & R/\alpha \le t \le R/\beta \\ 0 & t \ge R/\beta \end{cases}$$

Seismogram



falls off like $\frac{1}{R^2} \Rightarrow$ near field

P wave radiation pattern
from moment tensor source

$$\boxed{\text{ampl} \sim M_{ij}\gamma_i\gamma_j \sim \hat{r}\cdot M\cdot\hat{r}}$$

Fault source :

$$\mathrm{tr}\, \mathbf{M} = \int_\Sigma 2\mu\, \hat{\nu} \cdot \Delta s \; d\Sigma = 0$$

deviatoric $\qquad M = \frac{1}{3}(\mathrm{tr}\, \mathbf{M})\, \mathbf{I} + \mathbf{m}$

Consider $\hat{r} \cdot M \cdot \hat{r}$

$$= \underbrace{\hat{r} \cdot \hat{r}\, \tfrac{1}{3}(\mathrm{tr}\, M)}_{\substack{\text{isotropic} \\ \text{degree 0} \\ \text{spher. harm}}} + \underbrace{\hat{r} \cdot \mathbf{m} \cdot \hat{r}}_{\substack{\text{degree 2} \\ \text{spher harm}}}$$

$$\nabla^2 (M_{ij} x_i x_j) = M_{ii} = 0$$

$$\hat{r} \cdot M \cdot \hat{r} = Y_0 + \underbrace{Y_2}_{\text{only this for a fault source}}$$

Beachballs

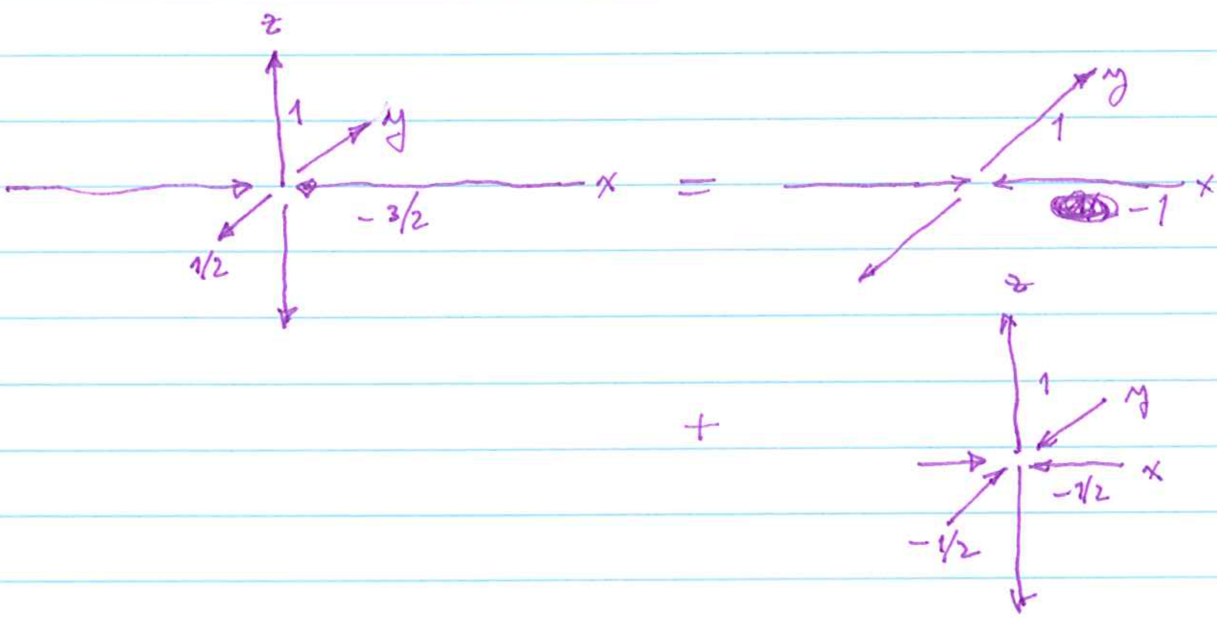
+ shaded

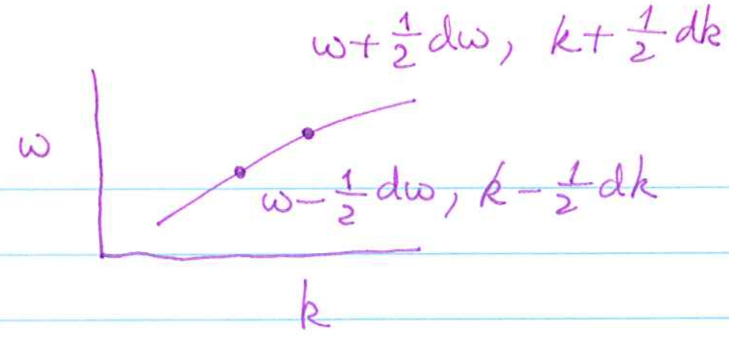
− unshaded

An example of the bfdc – clvd decomposition (5.17) – (5.19):

Say $m = -3/2$, $m' = 1$

$$\begin{pmatrix} -3/2 & & \\ & 1/2 & \\ & & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}}_{\text{bfdc}} + \begin{pmatrix} -1/2 & & \\ & -1/2 & \\ & & 1 \end{pmatrix}$$





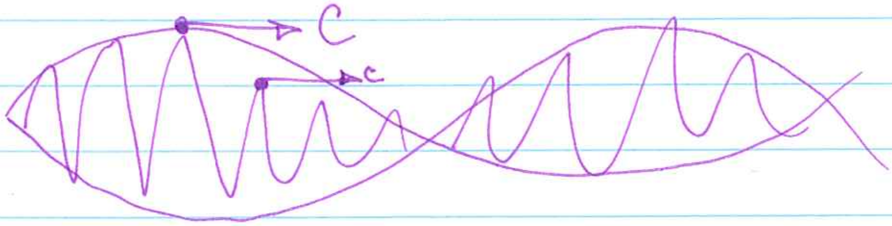
two neighboring waves

$$\cos\left[\left(\omega-\tfrac{1}{2}d\omega\right)t-\left(k-\tfrac{1}{2}dk\right)\Delta\right]$$

$$+\ \cos\left[\left(\omega+\tfrac{1}{2}d\omega\right)t+\left(k+\tfrac{1}{2}dk\right)\Delta\right]$$

$$=\ 2\cos\left(\omega t-k\Delta\right)\cos\left(d\omega\, t-dk\,\Delta\right)$$

beats



~~phase front~~ individual peaks and troughs travel with phase speed

$$\omega t-k\Delta=\text{constant}\quad(\text{zero, say})$$

$$\frac{\Delta}{t}=c=\frac{\omega}{k}\quad\text{phase speed}$$

envelope travels with group speed

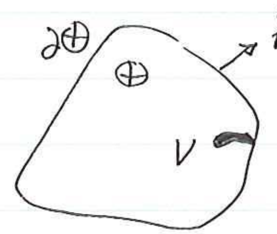
$$d\omega\, t-dk\,\Delta=0\qquad\frac{\Delta}{t}=C=\frac{d\omega}{dk}$$

Class #9

Stress glut $\underline{\underline{S}} = \underline{\underline{T}}^{Hooke} - \underline{\underline{T}}^{true}$

Equivalent body force $\underline{f} = -\nabla \cdot \underline{\underline{S}}$ in $\oplus$

Equivalent surface force $\underline{t} = \hat{n} \cdot \underline{\underline{T}}$ on $\partial\oplus$



source volume V

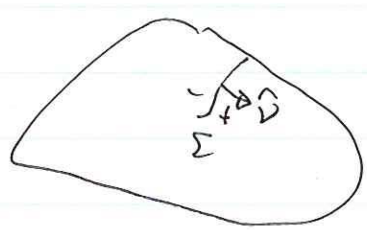
Total force $\underline{F}$ and torque $\underline{N}$ are zero (latter because $\underline{\underline{S}} = \underline{\underline{S}}^T$)

Acceleration response

$$\underline{a}(\underline{x}, t) = \sum_k (a_k \cos \omega_k t + b_k \sin \omega_k t)\, \underline{s}_k(\underline{x})$$

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \underbrace{\int_{t_0}^{t_f} \int_V}_{\substack{\text{integrate only over source volume } V \\ \text{and interval } t_0 \le t \le t_f}} \partial_t \underline{\underline{S}} : \underline{\underline{\varepsilon}}_k \begin{Bmatrix} \cos \omega_k t \\ \sin \omega_k t \end{Bmatrix} dV\, dt$$

Ideal fault



tang. slip $\underline{\Delta s}(\underline{x}, t) = \underline{s}(\underline{x}^+, t) - \underline{s}(\underline{x}^-, t)$

$\hat{\nu} \cdot \underline{\Delta s} = 0$

moment tensor density : $m_{ij} = c_{ijkl}\, \nu_j\, \Delta s_k$

$$\underline{\underline{m}} = \underline{\underline{c}} : \hat{\nu}\, \underline{\Delta s}$$

units $\dfrac{\text{moment}}{\text{area}} = \dfrac{\text{mass}}{\text{time}^2}$

Then $\quad \underline{\underline{S}} = \underline{\underline{m}}\,\delta_\Sigma$, confined to fault surface, depends only on slip $\underline{\Delta s}$ and orientation $\hat{\nu}$

$[\,\underline{\underline{C}}\,]^{\pm}$ across Σ

Equivalent body & surface force densities

$$\underline{f} = -\underline{\underline{m}}\cdot\nabla\delta_\Sigma \quad \text{in } \oplus$$

$$\underline{t} = (\hat{n}\cdot\underline{\underline{m}})\,\delta_\Sigma \quad \text{on } \partial\oplus$$

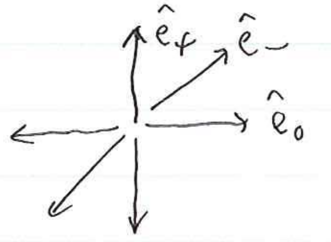
Completely general: dynamic time-dependent faulting in an anisotropic $\oplus$

Symmetry $\quad \underline{\underline{m}} = \underline{\underline{m}}^T \Rightarrow$ diagonalizable

$$\underline{\underline{m}} = m_+\hat{e}_+\hat{e}_+ + m_0\hat{e}_0\hat{e}_0 + m_-\hat{e}_-\hat{e}_-$$

Equivalent body force density
$$f_j = -m_{ij}\,\partial_j\delta_\Sigma$$

Three <u>linear</u> <u>vector</u> <u>dipoles</u>



$m_{+,0,-} > 0$: outward
$m_{+,0,-} < 0$: inward

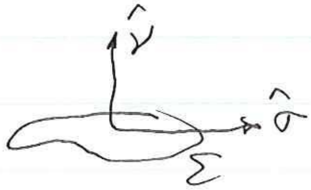
Isotropic $\oplus$: $C_{ijkl} = \overbrace{\left(\kappa - \frac{2}{3}\right)\delta_{ij}\delta_{kl}}^{\text{no contr. since } \nu_k \Delta s_k = 0} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

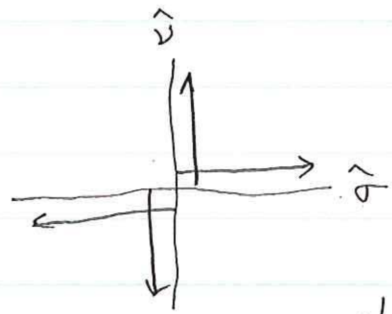
$$m_{ij} = \mu \Delta s \, (\nu_i \sigma_j + \sigma_i \nu_j)$$

where $\underline{\Delta s} = \Delta s \, \hat{\sigma}$

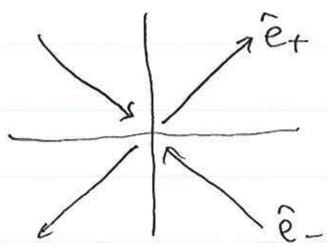
$$\boxed{\underline{\underline{m}} = \mu \Delta s \, (\hat{\nu}\hat{\sigma} + \hat{\sigma}\hat{\nu})}$$

Equivalent body force a <u>double couple</u>



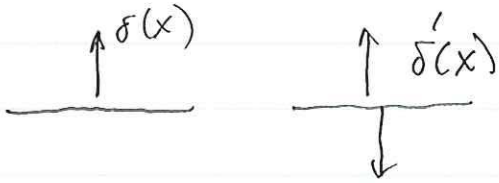


or



obviously no net force or torque

$$\underline{\underline{m}} = \mu \Delta s \, (\hat{e}_+ \hat{e}_+ - \hat{e}_- \hat{e}_-) \qquad \hat{e}_\pm = \frac{1}{\sqrt{2}}(\hat{\nu} \pm \hat{\sigma})$$



Valid for a <u>curved</u> or corrugated fault

Mode response to faulting

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \int_{t_0}^{t_f} \int_{\Sigma} \partial_t \underline{\underline{m}} : \underline{\underline{\varepsilon}}_k \begin{Bmatrix} \cos \omega_k t \\ \sin \omega_k t \end{Bmatrix} dV \, dt$$

integrate only ~~xxxx~~ over portion of Σ where slip is actively occurring.

<u>Point source</u> appro<u>ximation</u> : moment tensor

long wavelength >> source dimensions
long period >> source duration

$$\underline{\underline{\varepsilon}}_k(\underline{x}) \begin{Bmatrix} \cos \omega_k t \\ \sin \omega_k t \end{Bmatrix} \approx \underline{\underline{\varepsilon}}_k(\underline{x}_s) \begin{Bmatrix} \cos \omega_k t_s \\ \sin \omega_k t_s \end{Bmatrix}$$

x_s, t_s hypocenter location in point source approximation

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \underline{\underline{M}} : \underline{\underline{\varepsilon}}_k(\underline{x}_s) \begin{Bmatrix} \cos \omega_k t_s \\ \sin \omega_k t_s \end{Bmatrix}$$

where $M = \int_{t_0}^{t_f} \int_V \partial_t \underline{\underline{S}} \, dV \, dt$, <u>moment tensor</u>

$$= \int_V \underline{\underline{S}}^{\text{final state}} \, dV$$

volume integral of the final static stress glut.

$$\cos a \cos b + \sin a \sin b = \cos(a-b)$$

Response

$$\underline{a}(\underline{x},t) = \sum_k \underbrace{\underline{\underline{M}} : \underline{\underline{\varepsilon}}_k(x_s)}_{\text{ampl.}} \ \underbrace{\underline{s}_k(\underline{x})}_{\text{shape}} \ \underbrace{\cos \omega_k (t-t_s)}_{\text{begins at } t=t_s \text{ origin time}}$$

Linear in moment tensor $\underline{\underline{M}}$

Read quote from Gilbert

Moment tensor for an ideal earthquake fault

$$\underline{\underline{M}} = \int_{\Sigma} \underline{\underline{m}}^{\text{static final}} \, dA \qquad \left(\int_{\oplus} \underline{\underline{m}}^{\text{final}} \delta_{\Sigma} \, dV \right)$$

Isotropic $\oplus$:

$$\boxed{\underline{\underline{M}} = \int_{\Sigma} \mu \Delta s^{\text{final}} (\hat{\nu}\hat{\sigma} + \hat{\sigma}\hat{\nu}) \, dA}$$

Glut rate in point source approx

$$\partial_t \underline{\underline{S}} = \underline{\underline{M}} \, \delta(\underline{x} - \underline{x}_s) \, \delta(t - t_s)$$

Body force $\quad \underline{f} = - M \cdot \nabla\delta(\underline{x} - \underline{x}_s) \, \delta(t - t_s)$

3 linear vector dipoles

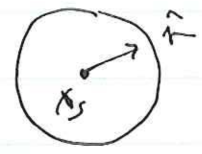
Planar fault, uni-directional slip :

$$\underline{\underline{M}} = M_0 (\hat{\nu}\hat{\sigma} + \hat{\sigma}\hat{\nu}) \qquad M_0 = \int_{\Sigma} \mu \Delta s^f \, dA$$

classic double couple $\qquad$ moment Aki (1966)

~~skip this~~

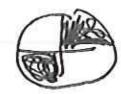
P-wave radiation pattern :

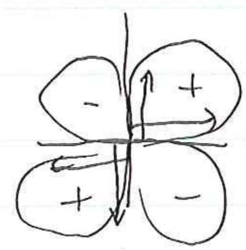


$\hat{r} \cdot \underline{\underline{M}} \cdot \hat{r}$

focal sphere

beachball





quadrupole pattern

talk about later

CMT = centroid - moment tensor

Solve for $\underline{\underline{M}}$ as well as an updated location $\underline{x}_s + \underline{\Delta x}$, $t_s + \Delta t$

$$\underline{a}(\underline{x},t) = \sum_k \underline{\underline{M}} : \underline{\underline{\varepsilon}}_k(\underline{x}_s)\, \underline{s}_k(\underline{x}) \cos \omega_k (t - t_s)$$

$$+ \sum_k \underline{\Delta x}\, \underline{\underline{M}} : \underline{\underline{\nabla \varepsilon}}_k(\underline{x}_s)\, s_k(\underline{x}) \cos \omega_k (t - t_s)$$

$$+ \sum_k \omega_k \Delta t\, \underline{\underline{M}} : \underline{\underline{\varepsilon}}_k(\underline{x}_s)\, \underline{s}_k(\underline{x}) \sin \omega_k (t - t_s)$$

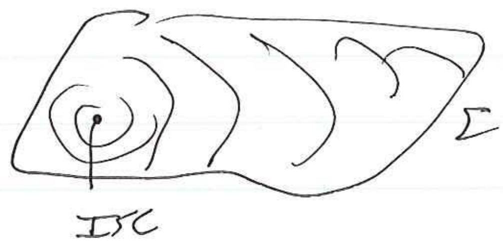
<u>Linear</u> inverse problem for $\underline{\underline{M}}$, $\underline{\Delta x}$, Δt

Can be shown (see DFT 5.4.2) that for a planar fault, uni-directional slip on fault

$$\underline{\Delta x} = \frac{1}{M_0}\int_\Sigma (\underline{x}-\underline{x}_s)\,\mu\,\Delta s^f\,dA$$

$$\Delta t = \frac{1}{M_0}\int_{t_0}^{t_f}\int_\Sigma (t-t_s)\,\mu\,\partial_t \Delta s\,dA\,dt$$

Centroid may differ from ISC location = beginning of rupture



Number of unknowns in a CMT inversion : 10

5.4.3 Deviatoric & double-couple sources

Can always decompose into isotropic + deviatoric

$$\underline{\underline{M}} = \frac{1}{3}(\mathrm{tr}\,\underline{\underline{M}})\,\underline{\underline{I}} + \underline{\underline{M}}^{dev}$$

ideal fault source in isotropic ⊕ : $\mathrm{tr}\,\underline{\underline{M}} = 0$

$$\underline{\underline{M}} = \int_\Sigma \mu\,\Delta s_f\,(\hat{\nu}\hat{\sigma} + \hat{\sigma}\hat{\nu})\,dA$$

curved fault

Common to impose linear constraint $\mathrm{tr}\,\underline{\underline{M}} = 0$
no volume change @ source

Some evidence for $tr\,\underline{\underline{M}} = 0$ mechanisms in volcanic settings

CMT unknowns $\rightarrow$ 9

May also wish to compare with best double couple

$\det \underline{\underline{M}}$ for a double couple

Non-linear : hard to impose

Instead CMT finds best-fitting double couple

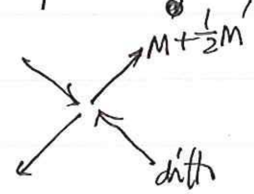
$$(\underline{\underline{M}}_{bfdc} - \underline{\underline{M}}) : (\underline{\underline{M}}_{bfdc} - \underline{\underline{M}}) = \min$$

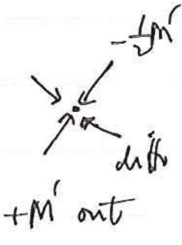
Solution found by diagonalization

$$\underline{\underline{M}} = \begin{pmatrix} M & & \\ & -M - M' & \\ & & M' \end{pmatrix} \quad \text{where } |M| \geq |M'|$$

note $tr\,\underline{\underline{M}} = 0$

$$= \underbrace{\begin{pmatrix} M + \frac{1}{2}M' & & \\ & -M - \frac{1}{2}M' & \\ & & 0 \end{pmatrix}}_{bfdc} + \underbrace{\begin{pmatrix} -\frac{1}{2}M' & & \\ & -\frac{1}{2}M' & \\ & & M' \end{pmatrix}}_{clvd}$$





CMT also gives so-called major and minor double couple

$$= \underbrace{\begin{pmatrix} M & & \\ & -M & \\ & & 0 \end{pmatrix}}_{\text{major}} + \underbrace{\begin{pmatrix} 0 & & \\ & -M' & \\ & & M' \end{pmatrix}}_{\text{minor}}$$

4% of events in CMT are significantly non-double-couple

Reasons: (1) noise in estimate
(2) anisotropy

$$M = \int_\Sigma \underline{\underline{c}} : \hat{\nu}\, \Delta s_f \, dA$$

probably small

(3) curvature of fault surface

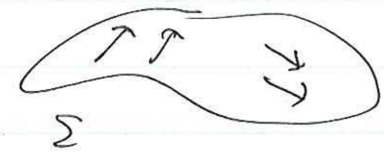
$$\underline{\underline{M}} = \int_\Sigma \mu \Delta s_f \, (\hat{\nu}\hat{\sigma} + \hat{\sigma}\hat{\nu})$$

Candidate geometries limited:
e.g. $\hat{\sigma} = \text{const}$

need not be whole can

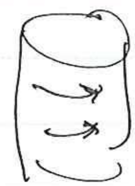
$$\underline{\underline{M}} = \int_\Sigma \mu \Delta s_f \, \hat{\nu} \, dA \, \hat{\sigma} + \hat{\sigma} \int_\Sigma \mu \Delta s_f \, \hat{\nu} \, dA$$

$\hat{\nu}$ constant



can do same

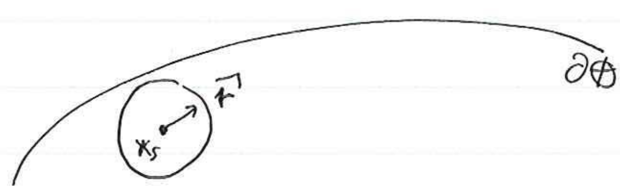
$\hat{\nu} \times \hat{r}$ constant



need not be whole can

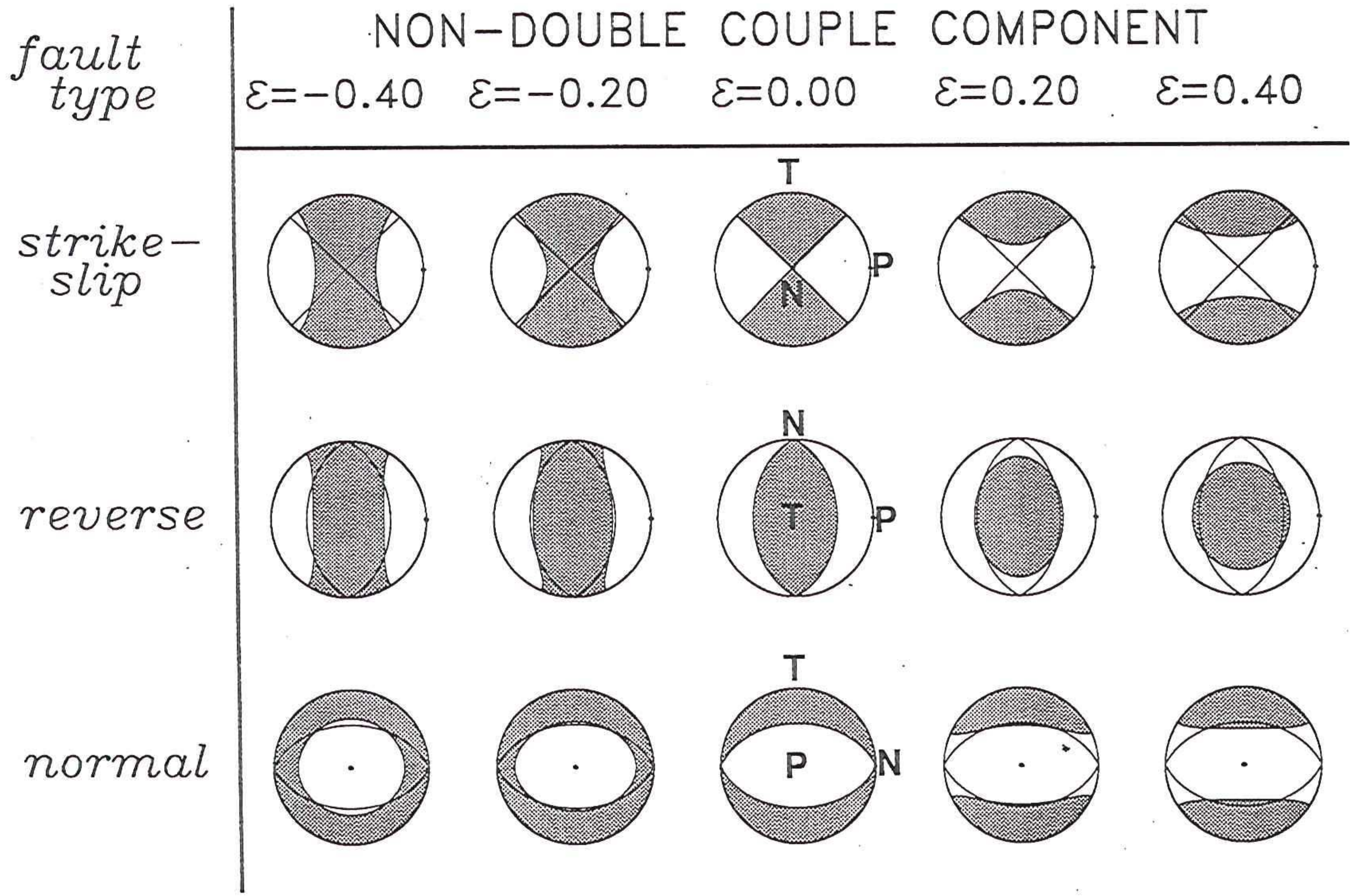
Most interesting — example — Göran Ekström's volcanic ring faults.

Beach balls $\hat{r} \cdot \underline{M} \cdot \hat{r}$

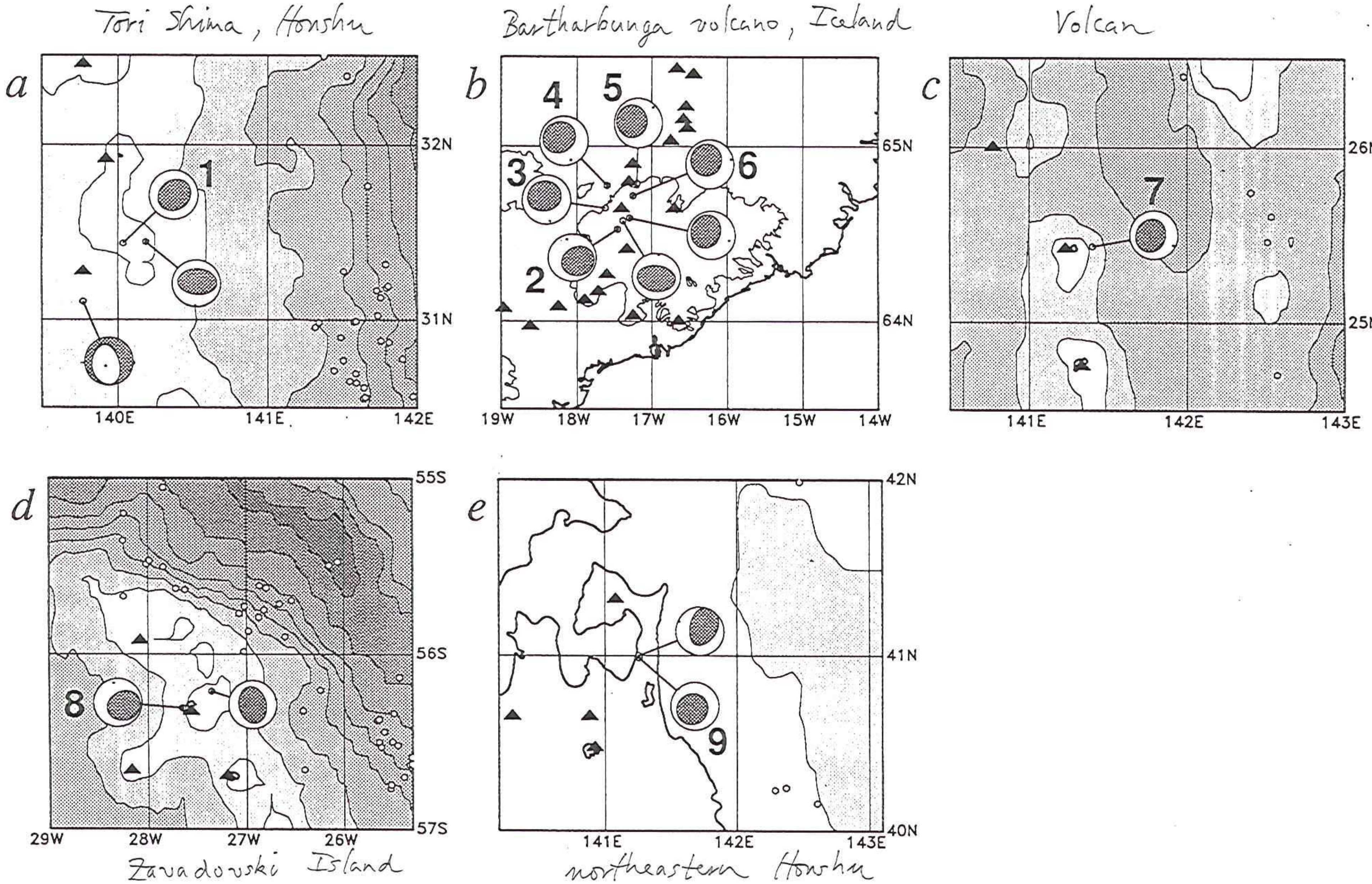


focal sphere

end class #9

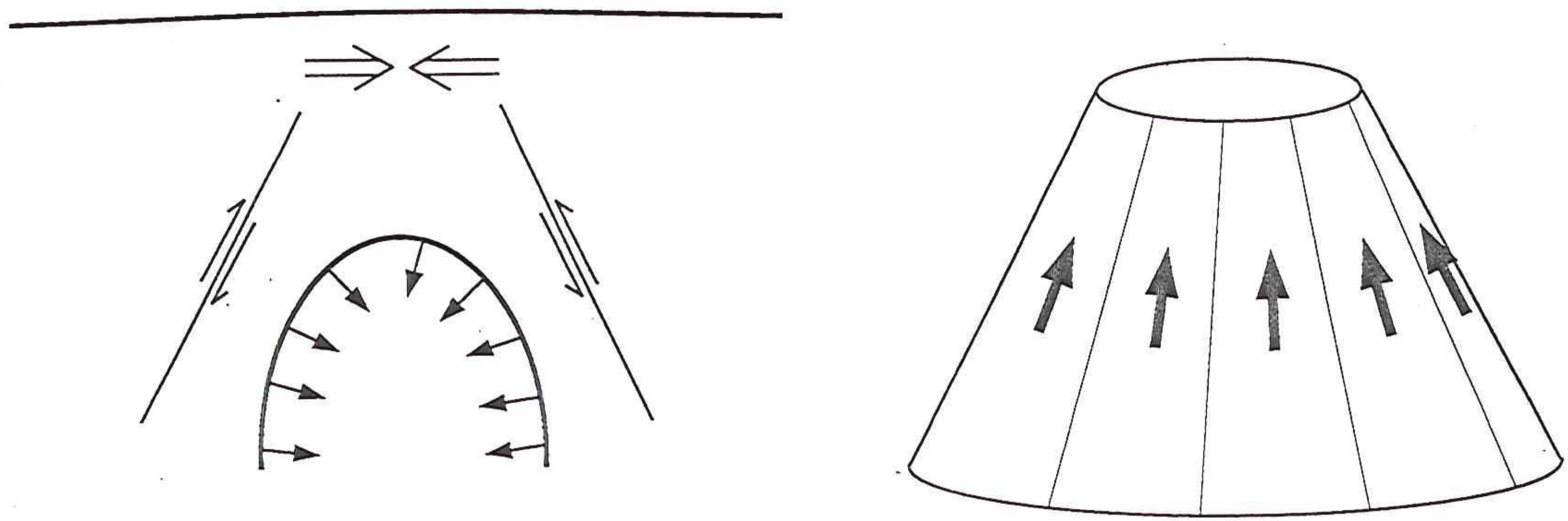


$$\varepsilon = \frac{-M_2}{\max(M_1, -M_3)}$$ where $M_1 \geq M_2 \geq M_3$

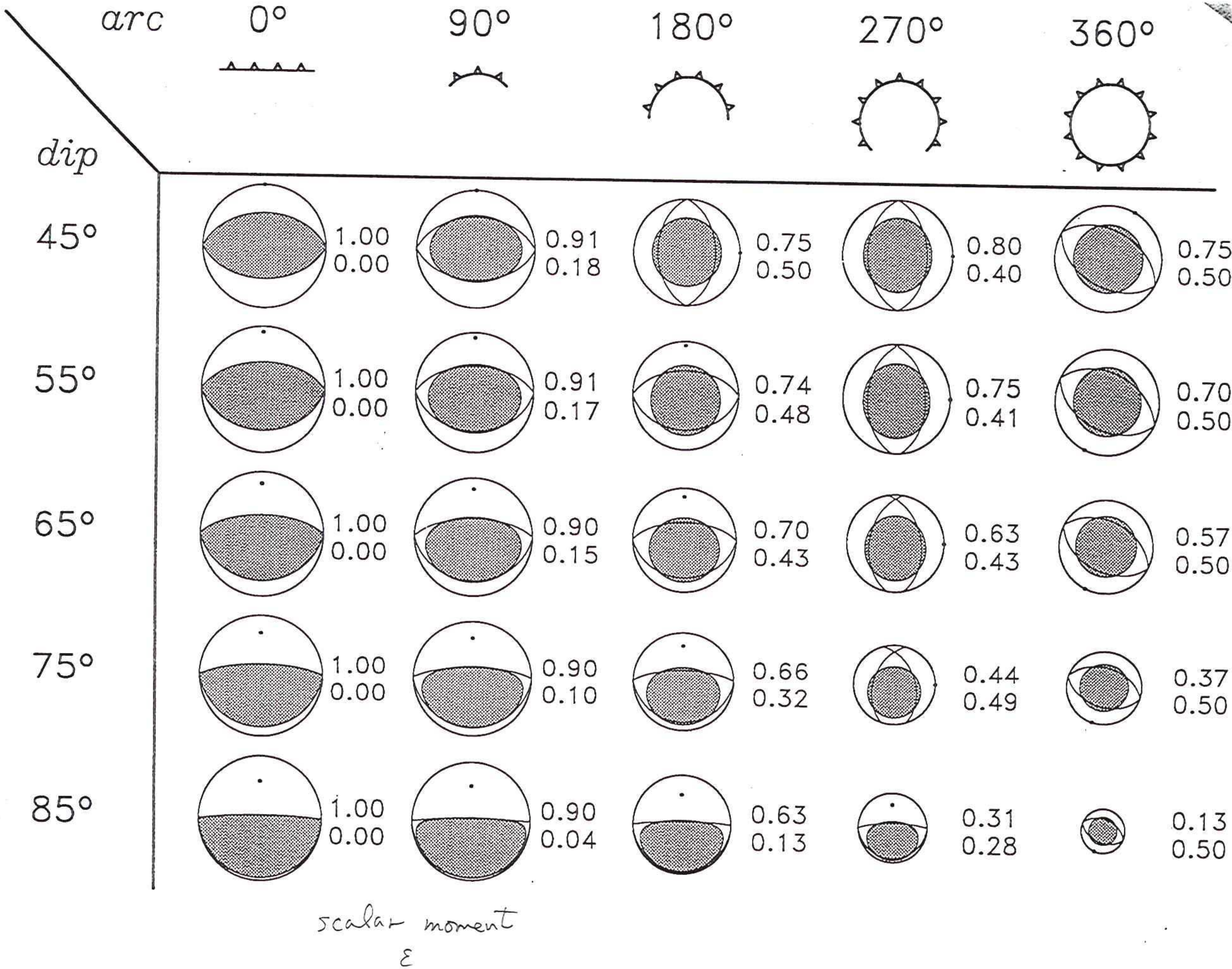


Tori Shima, Honshu
a
1
140E
141E
142E
32N
31N
Barthabunga volcano, Iceland
b
2
3
4
5
6
19W
18W
17W
16W
15W
14W
65N
64N
Volcan
c
7
141E
142E
143E
26N
25N
d
8
29W
28W
27W
26W
55S
56S
57S
Zavadovski Island
e
9
141E
142E
143E
42N
41N
40N
northeastern Honshu

a *b*



volcanic ring fault



arc
0°
90°
180°
270°
360°
dip
45°
55°
65°
75°
85°
1.00 0.00
0.91 0.18
0.75 0.50
0.80 0.40
0.75 0.50
1.00 0.00
0.91 0.17
0.74 0.48
0.75 0.41
0.70 0.50
1.00 0.00
0.90 0.15
0.70 0.43
0.63 0.43
0.57 0.50
1.00 0.00
0.90 0.10
0.66 0.32
0.44 0.49
0.37 0.50
1.00 0.00
0.90 0.04
0.63 0.13
0.31 0.28
0.13 0.50
scalar moment
ε

Class # 10

Eulerian vs Lagrangian

planets $\mathbf{r}_i(t)$, $i = 1, 2, \dots, N$

$\mathbf{r}(\mathbf{x}, t)$, $\mathbf{r}(\mathbf{x}, 0) = \mathbf{x}$ initial posn = particle label

$\mathbf{u}^L(\mathbf{x}, t) = \partial_t \mathbf{r}(\mathbf{x}, t)$

Eulerian = weather bureau

$\mathbf{r}$ = fixed in space

$\mathbf{u}^E(\mathbf{r}, t)$

Equivalent, complete descriptions

~~[illegible]~~ $\mathbf{u}^E(\mathbf{r}(\mathbf{x}, t), t) = \mathbf{u}^L(\mathbf{x}, t)$

Any physical variable, e.g. ϕ = temp, pressure, stress

$$\phi^E(\mathbf{r}(\mathbf{x}, t), t) = \phi^L(\mathbf{x}, t)$$

Chain rule: $\partial_t \phi^L = \partial_t \phi^E + \mathbf{u}^E \cdot \nabla_r \phi^E$
$= D_t \phi^E$

$D_t = \partial_t + \mathbf{u}^E \cdot \nabla_r$ material derivative

Take it for granted that you know the exact Eulerian conservation laws:

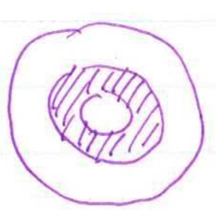
$$\partial_t \rho^E + \nabla \cdot (\rho^E \mathbf{u}^E) = 0 \qquad \text{continuity}$$
$$D_t \rho^E + \rho^E \nabla \cdot \mathbf{u}^E = 0$$

$$\rho^E D_t \mathbf{u}^E + \nabla \cdot \mathbf{T}^E = \rho^E \mathbf{g}^E \qquad \text{momentum}$$

gravity

Cauchy (symmetric)

SNREI Earth



$a = 6371$ km

undeformed state $\rho^0(r)$

grav. pot $\phi^0(r)$

$$\mathbf{g}^0 = -\nabla \phi^0$$

Poisson's eqn

$\nabla^2 \phi^0 = 4\pi \rho^0$ $\qquad G = 6.67 \times 10^{-11}\ \mathrm{N m^2 kg^{-2}}$

$$\mathbf{g}^0(\mathbf{r}) = -g^0(r)\,\hat{\mathbf{r}} \quad , \; g^0(r) = \partial_r \phi^0$$

$$\partial_r g^0 + 2r^{-1} g^0 = 4\pi G \rho^0$$

$$\frac{1}{r^2} \partial_r (r^2 g^0) = 4\pi G \rho^0$$

$$g^0(r) = 4\pi G r^{-2} \int_0^r \rho^0(r) r^2 \, dr$$
$$= G M_{\text{encl}}(r) / r^2$$

Static initial stress <u>hydrostatic</u>

$$\mathbf{T}^0(\mathbf{r}) = -p^0(r)\, \mathbf{I}$$

Mech. equil. $\rho^0 \nabla \phi^0 + \nabla \cdot \mathbf{T}^0 = 0$

$$\rho^0 \nabla \phi^0 + \nabla p^0 = 0$$ asphericity $\Rightarrow$ non-hydrostatic

Take curl: $\nabla \rho^0 \times \nabla \phi^0 = 0$

mult. by $\nabla p^0 \times$ $\nabla p^0 \times \nabla \phi^0 = 0$ level surfaces all coincide (spheres)

$$\partial_r p^0 + \rho^0 g^0 = 0$$

$$p^0(r) = \int_r^a \rho^0(r) g^0(r)\, dr$$ vanishes at $r = a$

Fig. 8.2 DFT: $p^0_{\oplus}$ center $= 364$ GPa

Now consider small oscillations of this model

$\mathbf{r}(\mathbf{x}, t) = \mathbf{x} + \underbrace{\mathbf{s}(\mathbf{x}, t)}_{\text{displacement}}$

density $\rho^0(r) + \rho^{E1}(\mathbf{r}, t)$

grav. pot. $\phi^0(r) + \phi^{E1}(\mathbf{r}, t)$

stress $-p^0(r)\mathbf{I} + \mathbf{T}^{E1}(\mathbf{r}, t)$

Continuity eqn $\partial_t(\rho^0 + \rho^{E1}) + \nabla \cdot [(\rho^0 + \rho^{E1})\mathbf{u}^E] = 0$

$$\partial_t \rho^{E1} + \nabla \cdot (\rho^0 \mathbf{u}^E) + \underbrace{\cancel{\nabla \cdot (\rho^{E1} \mathbf{u}^E)}}_{\text{ignore}} = 0$$

$$\mathbf{u}^E = D_t \mathbf{r} = D_t \mathbf{s} \approx \partial_t \mathbf{s}$$

integrate $\partial_t [\rho^{E1} + \nabla \cdot (\rho^0 \mathbf{s})] = 0$

$$\boxed{\rho^{E1} = -\nabla\cdot(\rho^0 \mathbf{s})} \leftarrow \text{1st order relation}$$

$$\rho^L(\mathbf{x},t) = \rho^E(\mathbf{r}(\mathbf{x},t),t)$$

$$\cancel{\rho^0} + \rho^{L1} = \cancel{\rho^0} + \mathbf{s}\cdot\nabla\rho^0 + \cdots + \rho^{E1} + \cdots$$

$$\boxed{\rho^{L1} = -\rho^0\, \nabla\cdot\mathbf{s}}$$

$$\rho^{L1} = \rho^{E1} + \underbrace{\mathbf{s}\cdot\nabla\rho^0}_{\text{advection term}}$$

Strictly speaking these eqns hold at $\mathbf{r}$ in deformed volume. But we regard as true at $\mathbf{x}$ in undeformed spherical volume. Then need to consider b.c. at boundaries

Change in grav pot due to redistr. of ⊕ mass

$$\nabla^2(\phi^0 + \phi^{E1}) = 4\pi G(\rho^0 + \rho^{E1})$$ exact at $\mathbf{r}$ in space

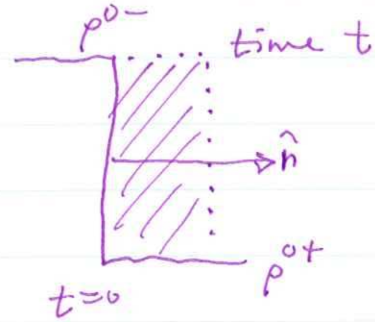
$$\boxed{\nabla^2\phi^{E1} = 4\pi G\rho^{E1}} \leftarrow$$ also interpret as valid at $\mathbf{x} \in$ undeformed

subject to b.c.

$[\phi^{E1}]^{+}_{-}$ on Σ

$$[\hat{n}\cdot\nabla\phi^{E1}]^{+}_{-} = -4\pi G\,[\rho^0]^{+}_{-}\,(\hat{n}\cdot s)$$

relative to $t=0$ $\exists$ a surface mass layer



surface mass density
$= \hat{n}\cdot s\,[\rho^{0-} - \rho^{0+}]$
> 0 in this picture
(outward displ. of CMB)

$$\phi^{E1}(x,t) = -G\int_{\oplus} \frac{\rho^{E1}(x',t)}{\|x-x'\|}\,d^3x' + G\int_{\Sigma} \frac{\overbrace{[\rho^0]^{+}_{-}}^{\text{at } x'?}(\hat{n}\cdot s)}{\|x-x'\|}\,d^2x'$$

$$= -G\int_{\oplus} \frac{\rho^0(x')\,s(x',t)\cdot(x-x')}{\|x-x'\|^3}\,d^3x'$$

Pert. in gravity $g^{E1} = -\nabla_x\phi^{E1}$

$$g^{E1}(x,t) = G\int_{\oplus} \rho^0(x')\,s(x',t)\cdot\Pi(x-x')\,d^3x'$$

$$\Pi = \frac{I}{\|x-x'\|^3} - \frac{3(x-x')(x-x')}{\|x-x'\|^5}$$

Deformation <u>everywhere</u> affects ϕ^{E1} and g^{E1} since gravity is a long-range force.

Momentum eqn

$$\underbrace{(\rho^0+\cancel{\rho^{E1}})}_{\text{ignore}}\underbrace{(D_t u^E)}_{\approx \partial_t^2 s} = \nabla\cdot(-p^0 I + T^{E1}) - (\rho^0+\rho^{E1})\nabla(\phi^0+\phi^{E1})$$

$$\rho^0\partial_t^2 s = \underbrace{\cancel{-\nabla p^0} + \nabla\cdot T^{E1} - \cancel{\rho^0\nabla\phi^0}}_{\text{equil.}} - \rho^0\nabla\phi^{E1} - \rho^{E1}\nabla\phi^0$$

$$\rho^0\partial_t^2 s = \nabla\cdot T^{E1} - \rho^0\nabla\phi^{E1} - \rho^{E1}\nabla\phi^0$$

Now we just set $T^{E1} = C : \nabla s = C : \varepsilon$ right.

WRONG!

Elastic constants must pertain to a parcel of matter; it is the Lagrangian perturbation T^{L1} not T^{E1} that is related to the strain by Hooke's law

Read quotes from Rayleigh (1906) and Love (1911) from page 6.

$$T^{L1} = T^{E1} + s\cdot\nabla T^0 = T^{E1} - (s\cdot\nabla p^0)I$$
$$= T^{E1} + (\rho^0 s\cdot\nabla\phi^0)I$$

Summarizing

$$* \quad \rho^0 \partial_t^2 s = -\rho^0 \nabla\phi^{E1} - \rho^{E1}\nabla\phi^0 - \nabla(\rho^0 s\cdot\nabla\phi^0) + \nabla\cdot T^{L1}$$

$$\rho^{E1} = -\nabla\cdot(\rho^0 s)$$ mixed Eulerian – Lagrangian description

$$\nabla^2\phi^{E1} = 4\pi G\rho^{E1}$$

$$T^{L1} = (\kappa - \tfrac{2}{3}\mu)(\nabla\cdot s)I + 2\mu\varepsilon$$

$*$ is an integro-differential eqn because $-\nabla\phi^{E1} = g^{E1}$ is an integral over whole $\oplus$.

In Ch. 8 we get rid of all subscripts

don't give these forms

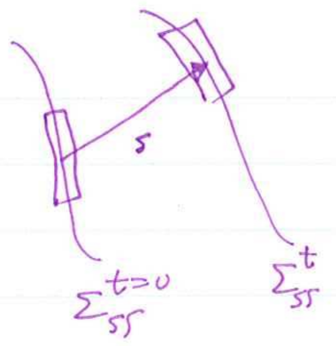
$$\rho\partial_t^2 s = -\rho\nabla\phi + \nabla\cdot(\rho s)\nabla\Phi - \nabla(\rho s\cdot\nabla\Phi) + \nabla\cdot T$$
$$\nabla^2\phi = -4\pi G\nabla\cdot(\rho s)$$
$$T = (\kappa - \tfrac{2}{3}\mu)(\nabla\cdot s)I + 2\mu\varepsilon$$

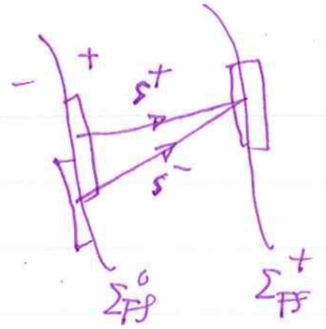
In a SNREI the momentum eqn is

$$\rho\partial_t^2 s = -\rho\nabla\phi - (4\pi G\rho^2 s_r)\hat{r} - \rho g[\nabla s_r - (\nabla\cdot s + \tfrac{2}{r}s_r)\hat{r}] + \nabla\cdot T = 0$$

Must be supplemented by b.c. on undeformed boundaries.

class 10 end here after 1.5 hours





Upshot: kinematic

$$[s]^+_- = 0 \quad \text{on } \Sigma_{SS}$$

$$[\hat{n}\cdot s]^+_- = 0 \quad \text{on } \Sigma_{FS}$$

grav.

$$[\phi^{E1}]^+_- = 0$$

$$[\hat{n}\cdot\nabla\phi^{E1} + 4\pi G\rho^0\hat{n}\cdot s]^+_- = 0 \quad \text{on } \Sigma$$

dynamic

$$[\hat{n}\cdot T^{L1}]^+_- = 0 \quad \text{on } \Sigma_{SS}$$

$$[\hat{n}\cdot T^{L1}]^+_- = \hat{n}\,[\hat{n}\cdot T^{L1}\cdot\hat{n}]^+_- = 0 \quad \text{on } \Sigma_{FS}$$

$$\hat{n}\cdot T^{L1} = 0 \quad \text{on } \partial V$$

Tables 3.3 and 3.4 pp. 103–104 D&T

Conservation of energy D&T Sect. 3.11.4

Take momentum eqn. Multiply by $\partial_t s$ and integrate over $\oplus$. Integrate by parts and apply the b.c.

No... wait... first talk about displacement versus displacement-potential points of view

Displacement : regard ⋆ as integro-differential

$-\nabla\phi^{E1} = g^{E1}$ given explicitly by

$$g^{E1} = G \int_{\oplus} \rho^0(x)\, s(x,t) \cdot \Pi(x-x')\, d^3x'$$

Solve only for s

Displacement-potential : treat ϕ^{E1} as an additional unknown. Then need and additional eqn $\nabla^2\phi^{E1} = 4\pi G\rho^{E1}$ plus associated b.c.

We very systematically do both — I will limit to displacement point of view in class

Now conservation of energy: DfT 3.11.4

$$\frac{d}{dt}\frac{1}{2}\int_{\oplus} [\underbrace{\rho^0\|\partial_t s\|^2}_{\text{kinetic}} + \underbrace{\varepsilon : C : \varepsilon}_{\text{elastic}} + \underbrace{\rho^0 s\cdot\nabla\phi^{E1}}_{\text{grav}}$$
$$+ \underbrace{\rho^0 s\cdot\nabla\nabla\phi^0\cdot s + \rho^0\nabla\phi^0\cdot(s\cdot\nabla s - s\nabla\cdot s)}_{\text{grav}}]\, dV = 0$$

This the conventional interpretation though not strictly correct. Actual elastic energy given by DfT 3.288 where J given by DfT 3.263. Actual gravitational energy (assemble from dispersal at ∞ into two states) given by DfT 3.223
Both elastic & grav. energy have a first order as well as 2nd order terms.

Look for normal mode solutions

$$\underline{s}(\underline{x}, t) = \underline{s}(\underline{x})\, e^{i\omega t}$$

$$\mathcal{H}\underline{s} = \omega^2 \underline{s}$$

$$\rho^0 \mathcal{H}\underline{s} = -\nabla\cdot\underline{\underline{T}}^{L1} + \nabla(\rho^0 \underline{s}\cdot\nabla\phi^0) + \rho^0\nabla\phi^{E1} + \rho^{E1}\nabla\phi^0$$

"together with the b.c.

Can show that $\mathcal{H}$ is Hermitian

$$\langle \underline{s}, \mathcal{H}\underline{s}' \rangle = \langle \mathcal{H}\underline{s}, \underline{s}' \rangle$$

Need to use that Σ is a level surface of ρ^0, ϕ^0, p^0.

Rayleigh's principle

$$\omega^2 = \frac{\mathcal{V}_e + \mathcal{V}_g}{\mathcal{T}} \quad \text{stationary}$$

$\mathcal{V}_e$, $\mathcal{V}_g$ given by D&T (4.168)–(4.169)

fractional elastic + grav. energies of a mode.

Cowling approximation